Polyadic Quantification without Identity

Exercises and Solutions

Part 2

1	$\exists x \forall y \neg F(x,y)$ /∴ $\exists x \forall y \forall z (F(x,z) \Rightarrow F(z,y))$	
2	$\exists x$	E C
3	$\forall y \neg F(x,y)$	E C, 1
4	$\forall y$	U C
5	$\forall z$	U C
6	$F(x,z)$ /∴ $F(z,y)$	
7	$\neg F(z,y)$ /∴ $\bot$	
8	$\forall y \neg F(x,y)$	Imp 3
9	$\neg F(x,z)$	UI, 8, 5
10	$\bot$	7, 9
11	$F(z,y)$	$\neg$ E, 7–10
12	$F(x,z) \Rightarrow F(z,y)$	$\Rightarrow$ I, 6–11
13	$\forall z (F(x,z) \Rightarrow F(z,y))$	U D
14	$\forall y \forall z (F(x,z) \Rightarrow F(z,y))$	U D
15	$\exists x \forall y \forall z (F(x,z) \Rightarrow F(z,y))$	$\exists$ D

1	$\forall x (F(x) \Leftrightarrow \forall y\, G(y))$ /∴ $\forall x\, F(x) \vee \forall x \neg F(x)$	
2	$\forall x ((F(x) \Rightarrow \forall y\, G(y)) \wedge (\forall y\, G(y) \Rightarrow F(x)))$	Equ. 1
3	$\forall x (F(x) \Rightarrow \forall y\, G(y)) \wedge \forall x (\forall y\, G(y) \Rightarrow F(x))$	QD
4	$\forall x (F(x) \Rightarrow \forall y\, G(y))$	$\wedge$ E, 3
5	$\forall x (\forall y\, G(y) \Rightarrow F(x))$	$\wedge$ E, 3
6	$\exists x\, F(x) \Rightarrow \forall y\, G(y)$	QD, 4 *
7	$\forall y\, G(y) \Rightarrow \forall x\, F(x)$	QD, 5 **
8	$\exists x\, F(x) \Rightarrow \forall x\, F(x)$	HS, 6, 7
9	$\neg \exists x\, F(x) \vee \forall x\, F(x)$	Impl., 8
10	$\forall x \neg F(x) \vee \forall x\, F(x)$	QN, 9
11	$\forall x\, F(x) \vee \forall x \neg F(x)$	Com., 10

Recall : * $\forall v (\mathcal{C}(v) \Rightarrow \alpha) \dashv\vdash \exists v\, \mathcal{C}(v) \Rightarrow \alpha$

** $\forall v (\alpha \Rightarrow \mathcal{C}(v)) \dashv\vdash \alpha \Rightarrow \forall v\, \mathcal{C}(v)$

where $v \notin FV[\alpha]$.

1	$\forall x\, \forall y\, \forall z\, ((F(x,y) \wedge F(y,z)) \Rightarrow F(x,z))$	
2	$\forall x\, \forall y\, (F(x,y) \Rightarrow F(y,x))$	
3	$\forall x\, F(x,x)$ $\therefore$ $\forall x\, \forall y\, \forall z\, ((F(x,y) \wedge F(x,z)) \Rightarrow F(y,z))$	
4	$\forall x$	
5	$\forall y$	
6	$\forall z$	
7	$F(x,y) \wedge F(x,z)$ $\therefore$ $F(y,z)$	
8	$F(x,y) \Rightarrow F(y,x)$	UC^2, 2
9	$F(x,y)$	$\wedge E$ 7
10	$F(y,x)$	$\Rightarrow E$, 8, 9
11	$\forall x\, \forall y\, \forall z\, ((F(x,y) \wedge F(y,z)) \Rightarrow F(x,z))$	Imp. 1
12	$\forall y\, \forall x\, \forall z\, ((F(y,x) \wedge F(x,z)) \Rightarrow F(y,z))$	RDV, 11
13	$(F(y,x) \wedge F(x,z)) \Rightarrow F(y,z)$	UI^3, 12, 4, 5, 6
14	$F(y,x) \wedge F(x,z)$	$\wedge I$, 9, 10
15	$F(y,z)$	$\Rightarrow E$, 13, 14
16	$(F(x,y) \wedge F(x,z)) \Rightarrow F(y,z)$	$\Rightarrow I$, 7–15
17	$\forall z\, ((F(x,y) \wedge F(x,z)) \Rightarrow F(y,z))$	UD
18	$\forall y\, \forall z\, ((F(x,y) \wedge F(x,z) \Rightarrow F(y,z))$	UD
19	$\forall x\, \forall y\, \forall z\, ((F(x,y) \wedge F(x,z) \Rightarrow F(y,z))$	UD

<u>Note</u>: line 8: 'UC^2' means UC twice
line 12: Interchange of x and y all done simultaneously.
line 13: 'UI^3' means UI done 3 times w.r.t. $\forall x$, $\forall y$, and $\forall z$.

1	$\forall x\,(F(x) \Rightarrow (G(x) \vee H(x)))$	
2	$\forall x\,((J(x) \wedge F(x)) \Rightarrow \neg G(x))$	
3	$\forall x\,(\neg F(x) \Rightarrow \neg J(x))$ $\quad /\therefore\ \forall x\,(J(x) \Rightarrow H(x))$	
4	$\forall x$	UC
5	$J(x)$ $\quad /\therefore\ H(x)$	
6	$F(x) \Rightarrow (G(x) \vee H(x))$	UC, 1
7	$(J(x) \wedge F(x)) \Rightarrow \neg G(x)$	UC, 2
8	$\neg F(x) \Rightarrow \neg J(x)$	UC, 3
9	$J(x) \Rightarrow F(x)$	Transp., DN, 8
10	$F(x)$	$\Rightarrow$E, 5, 9
11	$G(x) \vee H(x)$	$\Rightarrow$E, 6, 10
12	$J(x) \wedge F(x)$	$\wedge$I, 5, 10
13	$\neg G(x)$	$\Rightarrow$E, 7, 12
14	$H(x)$	DS, 11, 13
15	$J(x) \Rightarrow H(x)$	$\Rightarrow$I, 5–14
16	$\forall x\,(J(x) \Rightarrow H(x)$	UD

1	$\forall x\,\forall y\,\forall z\,((F(x,y) \wedge F(y,z)) \Rightarrow F(x,z))$	
2	$\neg \exists x\, F(x,x)$ $\quad /\therefore\ \forall x\,\forall y\,(F(x,y) \Rightarrow \neg F(y,x))$	
3	$\forall x\,\neg F(x,x)$	QN, 2
4	$\forall x$	UC
5	$\forall y$	UC
6	$F(x,y)$ $\quad /\therefore\ \neg F(y,x)$	
7	$\forall z\,((F(x,y) \wedge F(y,z)) \Rightarrow F(x,z))$	UC, 1
8	$\neg F(x,x)$	UC, 3
9	$(F(x,y) \wedge F(y,x)) \Rightarrow F(x,x)$	UI, 7, 4
10	$\neg (F(x,y) \wedge F(y,x))$	MT, 8, 9
11	$\neg F(x,y) \vee \neg F(y,x)$	DeM, 10
12	$\neg\neg F(x,y)$	DN, 6
13	$\neg F(y,x)$	DS, 11, 12
14	$F(x,y) \Rightarrow \neg F(y,x)$	$\Rightarrow$I, 6–13
15	$\forall y\,(F(x,y) \Rightarrow \neg F(y,x))$	UD
16	$\forall x\,\forall y\,(F(x,y) \Rightarrow \neg F(y,x))$	UD

1	$\forall x\,(F(x,x) \Rightarrow H(x))$	
2	$\exists x\, H(x) \Rightarrow \neg \exists y\, G(y)$ $\quad /\therefore\ \forall x\,(G(x) \Rightarrow \neg \exists z\, F(z,z))$	
3	$\quad \forall x$	UC
4	$\quad\quad G(x)$ $\quad /\therefore\ \neg \exists z\, F(z,z)$	
5	$\quad\quad \exists x\, G(x)$	EG, 4
6	$\quad\quad \exists y\, G(y)$	RDV, 5
7	$\quad\quad \exists x\, H(x) \Rightarrow \neg \exists y\, G(y)$	Imp. 2
8	$\quad\quad \neg\neg \exists y\, G(y)$	DN 6
9	$\quad\quad \neg \exists x\, H(x)$	MT, 7, 8
10	$\quad\quad \forall x\, \neg H(x)$	QN, 9
11	$\quad\quad \forall x\,(F(x,x) \Rightarrow H(x))$	Imp 1
12	$\quad\quad\quad \forall z$	UC
13	$\quad\quad\quad F(z,z) \Rightarrow H(z)$	UC, 11
14	$\quad\quad\quad \neg H(z)$	UC, 10
15	$\quad\quad\quad \neg F(z,z)$	MT, 13, 14
16	$\quad\quad \forall z\, \neg F(z,z)$	UD
17	$\quad\quad \neg \exists z\, F(z,z)$	QN, 16
18	$\quad G(x) \Rightarrow \neg \exists z\, F(z,z)$	$\Rightarrow$I, 4-17
19	$\forall x\,(G(x) \Rightarrow \neg \exists z\, F(z,z))$	UD

1	$\forall x\,(K(x) \Rightarrow (\exists y\, L(x,y) \Rightarrow \exists z\, L(z,x)))$	
2	$\forall x\,(\exists z\, L(z,x) \Rightarrow L(x,x))$	
3	$\neg \exists x\, L(x,x)$	$/\therefore\ \forall x\,(K(x) \Rightarrow \forall y\, \neg L(x,y))$
4	$\forall x\, \neg L(x,x)$	QN 3
5	$\forall x$	UC
6	$K(x) \Rightarrow (\exists y\, L(x,y) \Rightarrow \exists z\, L(z,x))$	UC, 1
7	$\exists z\, L(z,x) \Rightarrow L(x,x)$	UC, 2
8	$\neg L(x,x)$	UC, 4
9	$\neg \exists z\, L(z,x)$	MT, 8, 7
10	$K(x)$ $/\therefore\ \forall y\, \neg L(x,y)$	
11	$\exists y\, L(x,y) \Rightarrow \exists z\, L(z,x)$	$\Rightarrow$E, 6, 10
12	$\neg \exists z\, L(z,x)$	Imp. 9
13	$\neg \exists y\, L(x,y)$	MT, 11, 12
14	$\forall y\, \neg L(x,y)$	QN, 13
15	$K(x) \Rightarrow \forall y\, \neg L(x,y)$	$\Rightarrow$I, 10-14
16	$\forall x\,(K(x) \Rightarrow \forall y\, \neg L(x,y))$	UD

1.	$\neg \forall x\,(H(x) \vee K(x))$	
2	$\forall x\,((F(x) \vee \neg K(x)) \Rightarrow G(x,x))$	$/\therefore\ \exists x\, G(x,x)$
3	$\exists x\,(\neg H(x) \wedge \neg K(x))$	QN 1, DeM
4	$\exists x$	EC
5	$\neg H(x) \wedge \neg K(x)$	EC, 3
6	$(F(x) \vee \neg K(x)) \Rightarrow G(x,x)$	EC, 2
7	$\neg K(x)$	$\wedge$E, 5
8	$(F(x) \vee \neg K(x))$	$\vee$I, 7
9	$G(x,x)$	$\Rightarrow$E, 6, 8
10	$\exists x\, G(x,x)$	ED

1	$\exists x\, \forall y\, (\exists z\, F(y,z) \Rightarrow F(y,x))$		
2	$\forall x\, \exists y\, F(x,y)$ $\quad /\therefore\ \exists x\, \forall y\, F(y,x)$		
3	$\quad \exists x$	E C	
4	$\quad \forall y\, (\exists z\, F(y,z) \Rightarrow F(y,x))$	E C, 1	
5	$\quad \forall x\, \exists y\, F(x,y)$	Imp. 2	
6	$\quad \forall x\, \exists z\, F(x,z)$	R D V 5	*
7	$\quad \forall y\, \exists z\, F(y,z)$	R D V 6	**
8	$\quad\quad \forall y$	U C	
9	$\quad\quad \exists z\, F(y,z) \Rightarrow F(y,x)$	U C, 4	
10	$\quad\quad \exists z\, F(y,z)$	U C, 7	
11	$\quad\quad F(y,x)$	$\Rightarrow$ E, 9, 10	
12	$\quad \forall y\, F(y,x)$	U D	
13	$\exists x\, \forall y\, F(y,x)$	E D	

Could have done (*) and (**) in one step, i.e.,

i	$\forall x\, \exists y\, F(x,y)$		
	$\forall y\, \exists z\, F(y,z)$	R D V, i	$y/x,\ z/y$

1	$\exists x(F(x) \land \forall y(G(y) \Rightarrow H(y))$	
2	$\forall x(F(x) \Rightarrow (\neg L(x) \Rightarrow \neg\exists z(K(z) \land H(z))))$	$/\therefore\ \exists x(K(x) \land G(x)) \Rightarrow \exists x L(x)$
3	$\exists x$	EC
4	$F(x) \land \forall y(G(y) \Rightarrow H(y))$	EC, 1
5	$F(x) \Rightarrow (\neg L(x) \Rightarrow \neg\exists z(K(z) \land H(z)))$	EC, 2
6	$F(x)$	$\land$E, 4
7	$\forall y(G(y) \Rightarrow H(y))$	$\land$E, 4
8	$\neg L(x) \Rightarrow \neg\exists z(K(z) \land H(z))$	$\Rightarrow$E, 5, 6
9	$\exists x(\neg L(x) \Rightarrow \neg\exists z(K(z) \land H(z))$	ED
10	$\exists x\forall y(G(y) \Rightarrow H(y))$	ED
11	$\forall y(G(y) \Rightarrow H(y))$	EVDQ, 10
12	$\forall x(G(x) \Rightarrow H(x))$	RDV, 11
13	$\exists x(\exists z(K(z) \land H(z)) \Rightarrow L(x))$	Transp., 9, DN
14	$\exists z(K(z) \land H(z)) \Rightarrow \exists x L(x)$	Q Distr., 13 *
15	$\exists x(K(x) \land G(x))\ \ /\therefore\ \exists x L(x)$	
16	$\exists x$	EC
17	$K(x) \land G(x)$	EC, 15
18	$G(x) \Rightarrow H(x)$	EC, 12
19	$G(x)$	$\land$E, 17
20	$H(x)$	$\Rightarrow$E, 18, 19
21	$K(x)$	$\land$E, 17
22	$K(x) \land H(x)$	$\land$I, 20, 21
23	$\exists x(K(x) \land H(x))$	ED
24	$\exists z(K(z) \land H(z))$	RDV, 23
25	$\exists z(K(z) \land H(z)) \Rightarrow \exists x L(x)$	Imp 14
26	$\exists x L(x)$	$\Rightarrow$E, 24, 25
27	$\exists x(K(x) \land G(x)) \Rightarrow \exists x L(x)$	$\Rightarrow$I, 15–26

* Because of Quantifier Distribution previously proved. Recall:

$$\exists v(\alpha \Rightarrow Q(v)) \vdash\dashv (\alpha \Rightarrow \exists v\, Q(v))$$

where $v \notin FV[\alpha]$. In this case $x \notin FV[\exists z(K(z) \land H(z))]$

1	$\forall x(\exists y\, F(x,y) \Rightarrow \forall y\, F(y,x))$	
2	$\exists x \exists y\, F(x,y)$	$/\therefore\ \forall x \forall y\, F(x,y)$
3	$\quad \exists x$	EC
4	$\quad \exists y\, F(x,y) \Rightarrow \forall y\, F(y,x)$	EC, 1
5	$\quad \exists y\, F(x,y)$	EC, 2
6	$\quad \forall y\, F(y,x)$	$\Rightarrow$E, 4, 5
7	$\exists x \forall y\, F(y,x)$	ED
8	$\exists x \forall y\, F(y,x) \Rightarrow \forall y \exists x\, F(y,x)$	Theorem *
9	$\forall y \exists x\, F(y,x)$	$\Rightarrow$E, 7, 8
10	$\forall x \exists y\, F(x,y)$	RDV, 9 **
11	$\quad \forall x$	UC
12	$\quad \exists y\, F(x,y) \Rightarrow \forall y\, F(y,x)$	UC, 1
13	$\quad \exists y\, F(x,y)$	UC, 10
14	$\quad \forall y\, F(y,x)$	$\Rightarrow$E, 12, 13
15	$\forall x \forall y\, F(y,x)$	UD

* Recall: $\exists u \forall v\, Q(u,v) \Rightarrow \forall v \exists u\, Q(u,v)$ is a theorem.

** 9	$\forall y \exists x\, F(y,x)$	
9a	$\forall y \exists z\, F(y,z)$	RDV
9b	$\forall x \exists z\, F(x,z)$	RDV
10	$\forall x \exists y\, F(x,y)$	RDV

Steps 9a and 9b left out in proof.
You cannot go from 9 to $\forall y \exists y\, F(y,y)$, because that reduces simply to $\exists y\, F(y,y)$. Hence the usage of the variable z. However, one normally takes a shortcut. Given

$$\forall y \exists x\, F(y,x)$$

one <u>simultaneously</u> interchanges y with x.

1	$\forall x (M(x) \Rightarrow H(x))$	
2	$\exists x \exists y ((F(x) \wedge M(x)) \wedge (G(y) \wedge J(y,x)))$	
3	$\exists x\, H(x) \Rightarrow \forall y \forall z (\neg H(y) \Rightarrow \neg J(y,z))$	$/\therefore \exists x (G(x) \wedge H(x))$
4	$\exists x$	EC
5	$M(x) \Rightarrow H(x)$	EC 1
6	$\exists y ((F(x) \wedge M(x)) \wedge (G(y) \wedge J(y,x)))$	EC 2
7	$\exists y$	EC
8	$(F(x) \wedge M(x)) \wedge (G(y) \wedge J(y,x))$	EC 6
9	$M(x) \Rightarrow H(x)$	Imp 5
10	$F(x) \wedge M(x)$	$\wedge$E, 8
11	$G(y) \wedge J(y,x)$	$\wedge$E, 8
12	$M(x)$	$\wedge$E, 10
13	$H(x)$	$\Rightarrow$E, 9, 12
14	$\exists y\, H(x)$	ED
15	$H(x)$	EVDQ 14
16	$\exists x\, H(x)$	ED
17	$\forall y \forall z (\neg H(y) \Rightarrow \neg J(y,z))$	$\Rightarrow$E, 3, 16
18	$\exists x$	EC
19	$\exists y$	EC
20	$(F(x) \wedge M(x)) \wedge (G(y) \wedge J(y,x))$	EC 2 (2 time
21	$\forall z (\neg H(y) \Rightarrow \neg J(y,z))$	EC 17
22	$\neg H(y) \Rightarrow \neg J(y,x)$	UI, 21, 18
23	$J(y,x)$	$\wedge$E, 20, twice
24	$\neg\neg J(y,x)$	DN 23
25	$\neg\neg H(y)$	MT, 22, 24
26	$H(y)$	DN, 25
27	$G(y)$	$\wedge$E, 20, twice
28	$G(y) \wedge H(y)$	$\wedge$I, 26, 27
29	$\exists y (G(y) \wedge H(y))$	ED
30	$\exists x \exists y (G(y) \wedge H(y))$	ED
31	$\exists y (G(y) \wedge H(y))$	EVDQ 30
32	$\exists x (G(x) \wedge H(x))$	RDV 31

1	$\forall x(\exists y F(x,y) \Rightarrow \exists y \neg G(y))$	
2	$\exists x \exists y F(x,y)$	
3	$\forall x(G(x) \Leftrightarrow \neg H(x))$	$/\therefore \exists x H(x)$
4	$\underline{\exists x}$	EC
5	$\exists y F(x,y) \Rightarrow \exists y \neg G(y)$	EC, 1
6	$\exists y F(x,y)$	EC, 2
7	$\exists y \neg G(y)$	$\Rightarrow$ E, 5, 6
8	$\underline{\exists y}$	EC
9	$\neg G(y)$	EC, 7
10	$\forall x(G(x) \Leftrightarrow \neg H(x))$	Imp. 3
11	$G(y) \Leftrightarrow \neg H(y)$	UI, 10, 8
12	$(G(y) \Rightarrow \neg H(y)) \wedge (\neg H(y) \Rightarrow G(y))$	EQ, 11
13	$\neg H(y) \Rightarrow G(y)$	$\wedge$E, 12
14	$\neg\neg H(y)$	MT, 9, 13
15	$H(y)$	DN 14
16	$\exists y H(y)$	ED
17	$\exists x \exists y H(y)$	ED
18	$\exists y H(y)$	EVDQ, 17
19	$\exists x H(x)$	RDV, 18

1.	$\exists x(F(x) \wedge \forall y(G(y) \Rightarrow H(x,y)))$ /∴ $\exists x(F(x) \wedge (G(a) \Rightarrow H(x,a)))$	
2.	$\exists x$	EC
3.	$F(x) \wedge \forall y(G(y) \Rightarrow H(x,y))$	EC, 1
4.	$\forall y(G(y) \Rightarrow H(x,y))$	$\wedge$E, 3
5.	$G(a) \Rightarrow H(x,a)$	UI, 4
6.	$F(x)$	$\wedge$E, 3
7.	$F(x) \wedge (G(a) \Rightarrow H(x,a))$	$\wedge$I, 5, 6
8.	$\exists x(F(x) \wedge (G(a) \Rightarrow H(x,a)))$	ED

1.	$\forall x \forall y(G(x,y) \Leftrightarrow (F(y) \Rightarrow H(x)))$	
2.	$\forall z G(a,z)$ /∴ $\exists x F(x) \Rightarrow \exists x H(x)$	
3.	$\exists x F(x)$ /∴ $\exists x H(x)$	
4	$\forall y \forall x(G(y,x) \Leftrightarrow (F(x) \Rightarrow H(y)))$	DV, 1
5	$\forall x(G(a,x) \Leftrightarrow (F(x) \Rightarrow H(a)))$	UI, 4
6	$\exists u$	EC
7	$G(a,u)$	EC, 2
8	$G(a,u) \Leftrightarrow (F(u) \Rightarrow H(a))$	EC, 5
9	$F(u)$	EC, 3
10	$F(u) \Rightarrow H(a)$	$\Leftrightarrow$E, 7, 8
11	$H(a)$	$\Rightarrow$E, 9, 10
12	$\exists x H(x)$	EG, 11
13	$\exists u \exists x H(x)$	ED
14	$\exists x H(x)$	EVDQ 13
15	$\exists x F(x) \Rightarrow \exists x H(x)$	$\Rightarrow$I, 3 – 14.

1.	$\forall x(F(x) \Rightarrow \forall y(G(y) \Rightarrow H(x,y)))$	
2.	$\forall x(D(x) \Rightarrow \forall x(H(x,y) \Rightarrow C(y)))$ $/\therefore \exists x(F(x) \land D(x)) \Rightarrow \forall y(G(y) \Rightarrow C(y))$	
3.	$\exists x(F(x) \land D(x)$ $/\therefore \forall y(G(y) \Rightarrow C(y))$	
4.	$\exists x$	EC
5.	$F(x) \Rightarrow \forall y(G(y) \Rightarrow H(x,y))$	EC 1
6.	$D(x) \Rightarrow \forall y(H(x,y) \Rightarrow C(y))$	EC 2
7.	$F(x) \land D(x)$	EC 3
8.	$\forall y(H(x,y) \Rightarrow C(y))$	SL 6, 7
9.	$\forall y(G(y) \Rightarrow H(x,y))$	SL 5, 7
10.	$\forall y$	UC
11.	$G(y) \Rightarrow H(x,y)$	UC 9
12.	$H(x,y) \Rightarrow C(y)$	UC 8
13.	$G(y) \Rightarrow C(y)$	HS 11, 12
14.	$\forall y(G(y) \Rightarrow C(y))$	UD
15.	$\exists x \forall y(G(y) \Rightarrow C(y))$	ED
16.	$\forall y(G(y) \Rightarrow C(y))$	EVDQ 15
17.	$\exists x(F(x) \land D(x)) \Rightarrow \forall y(G(y) \Rightarrow C(y))$	$\Rightarrow$I, 3-16

1.	$\forall x((F(x) \lor H(x)) \Rightarrow (G(x) \land K(x)))$	
2.	$\neg \forall x(K(x) \land G(x))$ $/\therefore \exists x \neg H(x)$	
3.	$\exists x \neg(K(x) \land G(x))$	QN 2
4.	$\exists x$	EC
5.	$(F(x) \lor H(x)) \Rightarrow (G(x) \land K(x))$	EC 1
6.	$\neg(K(x) \land G(x))$	EC 3
7.	$\neg(G(x) \land K(x))$	Com 6
8.	$\neg(F(x) \lor H(x))$	MT 5, 7
9.	$\neg F(x) \land \neg H(x)$	DeM 8
10.	$\neg H(x)$	$\land$E, 9
11.	$\exists x \neg H(x)$	ED

$$\forall x\, \forall y\, (F(x,y) \Rightarrow \neg F(y,x)) \vdash \forall x\, \neg F(x,x)$$

1	$\forall x \forall y (F(x,y) \Rightarrow \neg F(y,x))$ $/\therefore$ $\forall x \neg F(x,x)$	
2	$\quad \forall x$	UC
3	$\quad\quad F(x,x)$ $/\therefore$ $\bot$	
4	$\quad\quad \forall y (F(x,y) \Rightarrow \neg F(y,x))$	UC 1
5	$\quad\quad (F(x,x) \Rightarrow \neg F(x,x))$	UI, 4, 2
6	$\quad\quad \neg F(x,x)$	
7	$\quad\quad \bot$	3, 6
8	$\quad \neg F(x,x)$	$\neg$I, 3 – 7
9	$\forall x \neg F(x,x)$	UD, 2 - 8

1	$\forall x \forall y (F(x,y) \Rightarrow F(y,x))$ $/\therefore$ $\forall x \forall y (F(x,y) \Leftrightarrow F(y,x))$	
2	$\quad \forall x$	UC
3	$\quad\quad \forall y$	UC
4	$\quad\quad F(x,y) \Rightarrow F(y,x)$	UC 2, 1
5	$\quad\quad\quad F(x,y)$ $/\therefore$ $F(y,x)$	
6	$\quad\quad\quad F(y,x)$	$\Rightarrow$E, 4, 5
7	$\quad\quad\quad F(y,x)$ $/\therefore$ $F(x,y)$	
8	$\quad\quad\quad \forall x \forall y (F(x,y) \Rightarrow F(y,x))$	Imp. 1
9	$\quad\quad\quad \forall u \forall v (F(u,v) \Rightarrow F(v,u))$	RDV, 8, $x \to u$, $y \to v$
10	$\quad\quad\quad \forall v (F(y,v) \Rightarrow F(v,y))$	UI 9, 3
11	$\quad\quad\quad F(y,x) \Rightarrow F(x,y)$	UI 10, 2
12	$\quad\quad\quad F(x,y)$	$\Rightarrow$E, 7, 11
13	$\quad\quad F(x,y) \Leftrightarrow F(y,x)$	$\Leftrightarrow$I, 5-6, 7-12
14	$\quad \forall y (F(x,y) \Leftrightarrow F(y,x))$	UD
15	$\forall x \forall y (F(x,y) \Leftrightarrow F(y,x))$	UD

Construct a formal proof of validity for each of the following arguments:

1. If there are any liberals, then all philosophers are liberals. If there are any humanitarians, then all liberals are humanitarians. So if there are any humanitarians who are liberals, then all philosophers are humanitarians. ($L(u)$: u is a liberal; $P(u)$: u is a philosopher; $H(u)$: u is a humanitarian)

1	$\exists x\, L(x) \Rightarrow \forall y\,(P(y) \Rightarrow L(y))$	
2	$\exists x\, H(x) \Rightarrow \forall y\,(L(y) \Rightarrow H(y))$ /∴ $\exists x\,(H(x) \wedge L(x)) \Rightarrow \forall y\,(P(y) \Rightarrow H(y))$	
3	$\exists x\,(H(x) \wedge L(x))$ /∴ $\forall y\,(P(y) \Rightarrow H(y))$	
4	$\exists x$	EC
5	$H(x) \wedge L(x)$	EC, 3
6	$H(x)$	$\wedge$E 4
7	$L(x)$	$\wedge$E 4
8	$\exists x\, L(x)$	ED
9	$\exists x\, H(x)$	ED
10	$\forall y\,(P(y) \Rightarrow L(y))$	$\Rightarrow$E, 1, 8
11	$\forall y\,(L(y) \Rightarrow H(y))$	$\Rightarrow$E, 1, 9
12	$\forall y$	UC
13	$P(y) \Rightarrow L(y)$	UC, 10
14	$L(y) \Rightarrow H(y)$	UC, 11
15	$P(y) \Rightarrow H(y)$	HS 13, 14
16	$\forall y\,(P(y) \Rightarrow H(y))$	UD
17	$\exists x\,(H(x) \wedge L(x)) \Rightarrow \forall y\,(P(y) \Rightarrow H(y))$	ED

2. There is a man whom all men despise. Therefore, at least one man despises himself. ($M(u)$: u is a man; $D(u,v)$: u despises v.)

1	$\exists x\,(M(x) \wedge \forall y\,(M(y) \Rightarrow D(y,x)))$ /∴ $\exists x\,(M(x) \wedge D(x,x))$	
2	$\exists x$	EC
3	$M(x) \wedge \forall y\,(M(y) \Rightarrow D(y,x))$	EC, 1
4	$M(x)$	$\wedge$E, 3
5	$\forall y\,(M(y) \Rightarrow D(y,x))$	$\wedge$E, 3
6	$M(x) \Rightarrow D(x,x)$	UI, 5, 2
7	$D(x,x)$	$\Rightarrow$E, 4, 6
8	$M(x) \wedge D(x,x)$	$\wedge$I, 4, 7
9	$\exists x\,(M(x) \wedge D(x,x))$	ED

3. All horses are animals. Therefore, the head of a horse is the head of an animal. ($E(u)$: u is a horse; $A(u)$: u is an animal; $H(u,v)$: u is the head of v.)

1	$\forall x\,(E(x) \Rightarrow A(x))$ /∴ $\forall x\,(\exists y\,(E(y) \wedge H(x,y)) \Rightarrow \exists y\,(A(y) \wedge H(x,y)))$	
2	$\forall x$	UC
3	$\exists y\,(E(y) \wedge H(x,y))$ /∴ $\exists y\,(A(y) \wedge H(x,y))$	
4	$\exists y$	EC
5	$E(y) \wedge H(x,y)$	EC, 3
6	$\forall x\,(E(x) \Rightarrow A(x))$	Imp. 1
7	$E(y) \Rightarrow A(y)$	UI, 6, 4
8	$E(y)$	$\wedge$E, 5
9	$A(y)$	$\Rightarrow$E, 7, 8
10	$H(x,y)$	$\wedge$E, 5
11	$A(y) \wedge H(x,y)$	$\wedge$I, 9, 10
12	$\exists y\,(A(y) \wedge H(x,y))$	ED
13	$\exists y\,(E(y) \wedge H(x,y)) \Rightarrow \exists y\,(A(y) \wedge H(x,y))$	$\Rightarrow$I, 3–12
14	$\forall x\,(\exists y\,(E(y) \wedge H(x,y)) \Rightarrow \exists y\,(A(y) \wedge H(x,y)))$	UD

__Note__: 'The head of a horse is the head of an animal' may be paraphrased as: 'All heads of horses are heads of animals'. Thus, $\forall x$((x is a head of a horse) $\Rightarrow$ (x is a head of an animal)). Thus, $\forall x\,(\exists y\,(E(y) \wedge H(x,y)) \Rightarrow \exists y\,(A(y) \wedge H(x,y)))$.

4. There is a professor who is liked by every student who likes at least one professor. Every student likes some professor or other. Therefore, there is a professor who is liked by all students. ($P(u)$: u is a professor; $S(u)$: u is a student; $L(u,v)$: u likes v.)

1	$\exists x\{P(x)\wedge\forall y\{[S(y)\wedge\exists z(P(z)\wedge L(y,z))]\Rightarrow L(y,x)\}\}$	
2	$\forall y[S(y)\Rightarrow\exists z(P(z)\wedge L(y,z))]$ /∴ $\exists x[P(x)\wedge\forall y(S(y)\Rightarrow L(y,x))]$	
3	$\quad\exists x$	EC
4	$\quad P(x)\wedge\forall y\{[S(y)\wedge\exists z(P(z)\wedge L(y,z))]\Rightarrow L(y,x)\}$	EC, 1
5	$\quad P(x)$	$\wedge$E, 4
6	$\quad\forall y\{[S(y)\wedge\exists z(P(z)\wedge L(y,z))]\Rightarrow L(y,x)\}$	$\wedge$E, 4
7	$\quad\quad\forall y$	UC
8	$\quad\quad\quad S(y)$ /∴ $L(y,x)$	
9	$\quad\quad\quad\forall y[S(y)\Rightarrow\exists z(P(z)\wedge L(y,z))]$	Imp. 2
10	$\quad\quad\quad S(y)\Rightarrow\exists z(P(z)\wedge L(y,z))$	UI, 9, 7
11	$\quad\quad\quad\exists z(P(z)\wedge L(y,z)$	$\Rightarrow$E, 8, 10
12	$\quad\quad\quad\forall y\{[S(y)\wedge\exists z(P(z)\wedge L(y,z))]\Rightarrow L(y,x)\}$	Imp., 6
13	$\quad\quad\quad[S(y)\wedge\exists z(P(z)\wedge L(y,z))]\Rightarrow L(y,x)$	UI, 12, 7
14	$\quad\quad\quad S(y)\wedge\exists z(P(z)\wedge L(y,z))$	$\wedge$I, 8, 11
15	$\quad\quad\quad L(y,x)$	$\Rightarrow$E, 13, 14
16	$\quad\quad S(y)\Rightarrow L(y,x)$	$\Rightarrow$I, 8–15
17	$\quad\forall y(S(y)\Rightarrow L(y,x))$	UD
18	$\quad P(x)\wedge\forall y(S(y)\Rightarrow L(y,x))$	$\wedge$I, 5, 17
19	$\exists x[P(x)\wedge\forall y(S(y)\Rightarrow L(y,x))]$	ED

5. No one respects a person who does not respect himself. No one will hire a person he does not respect. Therefore, a person who respects no one will never be hired by anybody. ($P(u)$: u is a person; $R(u,v)$: u respects v; $H(u,v)$: u hires v.)

1	$\forall x[(P(x) \wedge \neg R(x,x)) \Rightarrow \forall y(P(y) \Rightarrow \neg R(y,x))]$	
2	$\forall y\{P(y) \Rightarrow \forall x[(P(x) \wedge \neg R(y,x)) \Rightarrow \neg H(y,x)]\}$ $/\therefore \forall x\{[P(x) \wedge \forall z(P(z) \Rightarrow \neg R(x,z))] \Rightarrow \forall y(P(y) \Rightarrow \neg H(y,x))\}$	
3	$\forall x$	UC
4	$P(x) \wedge \forall z(P(z) \Rightarrow \neg R(x,z))$ $/\therefore \forall y(P(y) \Rightarrow \neg H(y,x))$	
5	$\forall y$	UC
6	$P(y)$ $/\therefore \neg H(y,x)$	
7	$\forall z(P(z) \Rightarrow \neg R(x,z))$	$\wedge$E, 4 and Imp.
8	$P(x) \Rightarrow \neg R(x,x)$	UI, 7, 3
9	$P(x)$	$\wedge$E, 4 and Imp.
10	$\neg R(x,x)$	$\Rightarrow$E, 8, 9
11	$P(x) \wedge \neg R(x,x)$	$\wedge$I, 9, 10
12	$\forall x[(P(x) \wedge \neg R(x,x)) \Rightarrow \forall y(P(y) \Rightarrow \neg R(y,x))]$	Imp. 1
13	$(P(x) \wedge \neg R(x,x)) \Rightarrow \forall y(P(y) \Rightarrow \neg R(y,x))$	UI, 12, 3
14	$\forall y(P(y) \Rightarrow \neg R(y,x))$	$\Rightarrow$E, 11, 13
15	$P(y) \Rightarrow \neg R(y,x)$	UI, 14, 5
16	$\neg R(y,x)$	$\Rightarrow$E, 6, 15
17	$P(x) \wedge \neg R(y,x)$	$\wedge$I, 9, 16
18	$\forall y\{P(y) \Rightarrow \forall x[(P(x) \wedge \neg R(y,x)) \Rightarrow \neg H(y,x)]\}$	Imp. 2
19	$P(y) \Rightarrow \forall x[(P(x) \wedge \neg R(y,x)) \Rightarrow \neg H(y,x)]$	UI, 18, 5
20	$\forall x[(P(x) \wedge \neg R(y,x)) \Rightarrow \neg H(y,x)]$	$\Rightarrow$E, 6, 19
21	$(P(x) \wedge \neg R(y,x)) \Rightarrow \neg H(y,x)$	UI, 20, 3
22	$\neg H(y,x)$	$\Rightarrow$E, 17, 21
23	$P(y) \Rightarrow \neg H(y,x)$	$\Rightarrow$I, 6–22
24	$\forall y(P(y) \Rightarrow \neg H(y,x))$	UD
25	$[P(x) \wedge \forall z(P(z) \Rightarrow \neg R(x,z))] \Rightarrow \forall y(P(y) \Rightarrow \neg H(y,x))$	$\Rightarrow$I, 4-24
26	$\forall x\{[P(x) \wedge \forall z(P(z) \Rightarrow \neg R(x,z))] \Rightarrow \forall y(P(y) \Rightarrow \neg H(y,x))\}$	UD

6. Alfred shaves all and only those inhabitants of Berkeley who do not shave themselves. Alfred is an inhabitant of Berkeley. Therefore, Alfred does not shave himself. ($F(u,v)$: u is an inhabitant of v; $S(u,v)$: u shaves v; a: Alfred; b: Berkeley.)

1	$\forall x\,(F(x,b) \Rightarrow (S(a,x) \Leftrightarrow \neg S(x,x)))$	
2	$F(a,b)$ /∴ $\neg S(a,a)$	
3	$S(a,a)$ /∴ $\perp$	
4	$F(a,b) \Rightarrow (S(a,a) \Leftrightarrow \neg S(a,a))$	UI, 1
5	$S(a,a) \Leftrightarrow \neg S(a,a)$	
6	$\neg S(a,a)$	
7	$\perp$	
8	$\neg S(a,a)$	

7. All cars are useful. Therefore, everyone who has a car has something useful. ($P(u)$: u is a car; $Q(u)$: u is useful; $R(u)$: u is a person; $S(u,v)$: u has v.) Domain: all objects.

1	$\forall x\,(P(x) \Rightarrow Q(x))$ /∴ $\forall y \forall x\,[((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists z\,(Q(z) \wedge S(y,z))]$	
2	$\forall y$	
3	$\forall x$	
4	$(R(y) \wedge S(y,x)) \wedge P(x)$ /∴ $\exists x\,(Q(z) \wedge S(y,z))$	
5	$P(x)$	$\wedge$E, 4
6	$\forall x\,(P(x) \Rightarrow Q(x))$	Imp. 1
7	$P(x) \Rightarrow Q(x)$	UI, 6, 3
8	$R(y) \wedge S(y,x)$	$\wedge$E, 4
9	$S(y,x)$	$\wedge$E, 8
10	$Q(x)$	$\Rightarrow$E, 5, 7
11	$Q(x) \wedge S(y,x)$	$\wedge$I, 9, 10
12	$\exists x\,(Q(x) \wedge S(y,x))$	EG, 11
13	$\exists z\,(Q(z) \wedge S(y,z))$	RDV, 12
14	$((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists z\,(Q(z) \wedge S(y,z))$	$\Rightarrow$I 4-13
15	$\forall x\,[((R(y) \wedge S(y,x)) \wedge P(x)) \Rightarrow \exists z\,(Q(z) \wedge S(y,z))]$	UD
16	$\forall y \forall x\,[((R(y) \wedge S(y,x)) \wedge P(x) \Rightarrow \exists z\,(Q(z) \wedge S(y,z))]$	UD