Philosophy 352

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Contents

The Formal Syntax for the Language of Quantification Logic

1 The Alphabet of Quantification Logic (QL)

The *alphabet* of QL consists of the following symbols:

1. **Zero-Place Predicates**:

$$\mathcal{S} := \{P_1^0, P_2^0, \ldots\} \tag{1}$$

Remark 1.1 *We will also allow* $\{A, B, \ldots\}$ *(with or without upper and/ or lower indices) to serve as zero-place predicate symbols.*

Remark 1.2 *Zero-place predicate symbols, also called zero-place relations, correspond to the* atomic *sentence letters of sentence logic SL. We admit the atomic sentence symbols of SL as part of the alphabet of QL only because many textbooks do. However, strictly speaking, atomic sentence symbols do not make sense in QL. Consider the sentence 'Socrates is mortal'. In SL we would simply symbolize that sentence as M because the sentence 'Socrates is mortal' is atomic. That is, in SL we ignore the predicative structure of 'Socrates is mortal'. Now in QL, the predicative structure should never be ignored even in the case where the sentence in question does not contain any quantifying phrases such as 'for all' or 'there exists'. So, strictly speaking, the sentence 'Socrates is mortal', which does not contain any quantifying phrases, should be symbolized as $M(a)$, where M is the monadic (one-place) predicate symbol standing for 'being mortal' and a is the individual constant symbol (name symbol) standing for the name 'Socrates'. So strictly speaking, zero-place predicate symbols $A, B, \ldots$ have no meaning in QL and should therefore not be part of QL's alphabet. We suspect that many textbooks include zero-place predicates into the alphabet of QL in order to simplify things. However, we see no real virtue in simplifying matters at the expense of conceptual clarity and consistency.*

2. **Predicate Symbols**:
$$\mathcal{P} := \{P^1_1, P^1_2, \ldots; P^2_1, P^2_2, \ldots; P^3_1, P^3_2 \ldots; \ldots\} \tag{2}$$

   Remark 1.3 *We will also allow $\{A, B, C, \ldots\}$ (with or without upper and/or lower indices) to serve as predicate symbols.*

   Remark 1.4 *Note that* Identity *('=') will be regarded as a 2-place predicate. For example, P^2_1 might stand for '='.*

3. **Function Symbols**:
$$\mathcal{F} := \{f^1_1, f^1_2, \ldots; f^2_1, f^2_2, \ldots; f^3_1, f^3_2, \ldots; \ldots\} \tag{3}$$

4. **Individual Constants (Designators)**:
$$\mathcal{C} := \{a_1, a_2, \ldots\} \tag{4}$$

   Remark 1.5 *The individual constant symbols are used as names of objects in the universe of discourse UD.*

   We shall also use the letters $\{a, b, c, d, e\}$[1].

5. **Individual Variables**:
$$\mathcal{V} := \{x_1, x_2, \ldots\} \tag{5}$$

   Remark 1.6 *In a more informal context we shall also simply use x, y, z as variables.*

6. **Logical Operators**:
$$\mathcal{O} := \{\neg, \wedge, \vee, \Longrightarrow, \Longleftrightarrow, \perp, \forall, \exists\} \tag{6}$$

7. **Auxiliary Symbols**: ')', '(', ','.

Remark 1.7 *Metavariables*

[1]Note that the *Logic Book* allows all the lower case Roman letters 'a' through 'v'. This is inconvenient for us since we shall use some letters such as 'f', 'g', 'r', 's', 't', 'u' and 'v' to serve other roles. In the *Logic Book* the syntax of QL does not include function terms; consequently there is no need for the *Logic Book* to use the symbol 'f' for that purpose. However, we will go along with the textbook and use additional lower case letters from the English alphabet provided no confusion arises as a consequence.

1. *We will use gothic '$\mathfrak{a}$', with or without subscript, as a metavariable ranging over the individual constant symbols of QL. That is,*

$$\mathfrak{a} \in \mathcal{C} := \{a, b, c, d, e, \dots; a_1, a_2, \dots\}. \tag{7}$$

2. *We will use gothic '$\mathfrak{f}$', with or without subscripts and/or superscripts, as a metavariable ranging over the set of function symbols. That is,*

$$\mathfrak{f} \in \mathcal{F} := \{f_1^1, f_2^1, \dots; f_1^2, f_2^2, \dots; f_1^3, f_2^3, \dots; \dots\}. \tag{8}$$

3. *We will use u, v, w, with or without subscripts, as meta-variables ranging over the individual variables of QL. That is,*

$$u, v, w \in \mathcal{V} := \{x, y, z; x_1, x_2, \dots, \}. \tag{9}$$

2 Terms and Well-Formed Formulas of QL

To specify a formal language of QL completely, we need in addition to its alphabet an effective procedure for deciding (constructing) which sequences of symbols of the alphabet constitute the wffs of QL. However, before we can recursively (inductively) define the set of wffs of QL, we first need to define what a *term* is.

2.1 The Terms of QL

Definition 2.1 (Terms) *Let $\mathcal{T}$ denote the set of all terms in QL. We define $\mathcal{T}$ recursively as follows:*

1. *Individual variables $\{x_1, x_2, \dots\}$ are terms.*
2. *Individual constants (designators) $\{a_1, a_2, \dots\}$ are terms.*
3. *If $\mathfrak{f}_i^n$ is a function symbol in QL, and if $t_1, \dots, t_n \in \mathcal{T}$, then $\mathfrak{f}_i^n(t_1, \dots, t_n) \in \mathcal{T}$.*
4. *Nothing is in $\mathcal{T}$ unless it is generated by (1), (2) and (3).*

Remark 2.2 *We will also write variables without subscripts using simply x, y, z. The same goes for constant symbols and function symbols which may respectively be written as a, b, d, e and f, g, h.*

Remark 2.3 *Terms are those expressions in QL which will be interpreted as objects, that is,*

1. *as the things to which functions are applied*
2. *as the things which have properties*
3. *or as the things about which assertions are made.*

The most important kind of terms in logic and mathematics are variables.

Example 2.4 *The following are all terms as can easily be checked by applying definition 2.1.*

$$f_1^1(x_5) \tag{10}$$
$$f_1^2(x_1, x_2) \tag{11}$$
$$f_1^3(x_1, x_4, f_1^2(x_1, x_2)) \tag{12}$$
$$f_2^3(a_{10}, x_1, f_1^2(x_1, x_2)) \tag{13}$$
$$f_1^4(f_1^1(x_5), a_3, f_1^2(x_1, a_2), f_2^3(a_{10}, x_1, f_1^2(x_1, x_2))). \tag{14}$$

Example 2.5 *The definition for terms of QL describes how terms are constructed in QL from constants and variables using function symbols. Each term of QL can be constructed, by means of function symbols, by starting with the basic terms, namely constants and variables of QL. Consider the term*

$$\sqrt{x^2 + y^3 - 9} \tag{15}$$

Let the functions $f_1^1, f_2^1, f_3^1, f_4^3, f_5^1$ *be defined as follows:* $f_1^1(t) = t^2$, $f_2^1(t) = t^3$, $f_3^1(t) = t$, $f_4^3(t_1, t_2, t_3) = t_1 + t_2 - t_3$ *and* $f_5^1(t) = \sqrt{t}$. *The basic terms are* $\{x, y, 9\}$. *Using functions* $\{f_1^1, f_2^1, f_3^1\}$ *respectively on the basic terms* $\{x, y, 9\}$ *we get the terms* $\{x^2, y^3, 9\}$. *Applying the function* f_4^3 *to those terms gives*

$$f_4^3(t_1, t_2, t_3) = x^2 + y^3 - 9. \tag{16}$$

And lastly, applying f_5^1 *to the term* $x^2 + y^3 - 9$ *yields the term* $\sqrt{x^2 + y^3 - 9}$.

Remark 2.6 *It should be noted that the functions in example 2.5 were defined in terms of symbols such as '+', '−' and* x^2 *etc. which are not strictly speaking part of QL's alphabet. We could write the term* $\sqrt{x^2 + y^3 - 9}$ *more formally as*

$$f_5^1\left(f_4^3\left(f_1^1(x), f_2^1(y), f_3^1(9)\right)\right) \tag{17}$$

Clearly, all the symbols of (17) are in QL's alphabet. While formally more correct, this way of writing things is very cumbersome and we shall often, in fact most often, express things more informally.

2.2 Atomic Formulas of QL

Definition 2.7 (Atomic Wffs) *Let* $\mathcal{AF}$ *denote the set of all atomic formulas of QL and let* $\mathcal{S}$ *denote the atomic sentence letters of SL. Then*

1. $\bot \in \mathcal{AF}$
2. $\varphi_i \in \mathcal{S}$ *then* $\varphi_i \in \mathcal{AF}$.
3. *If* $t_1, t_2 \in \mathcal{T}$ *then* $(t_1 = t_2) \in \mathcal{AF}$
4. *If* $t_1, \ldots, t_n \in \mathcal{T}$ *and* $\varphi_i^n \in \mathcal{P}$, *then* $\varphi_i^n(t_1, \ldots, t_n) \in \mathcal{AF}$
5. *Nothing is an atomic formula of QL unless it is generated by (1)–(4).*

Remark 2.8 *We introduce a special-purpose atomic wff* $\bot$ *into QL which we call the* **Falsum**. *It is also sometimes referred to as* absurdity. *Think of* $\bot$ *as standing for any logically absurd situation such as* $0 = 1$ *or* $\alpha \wedge \neg\alpha$. *Indeed one might from a semantic point of view say that* $\bot$ *represents or stands for any truth-functionally inconsistent set of wffs. If a truth-functionally inconsistent set of wffs has only one member then it follows from the definition of truth-functional inconsistency that the single wff contained in the set must be truth-functionally false.*

From a syntactic point of view $\bot$ *represents any set of syntactically inconsistent wffs.*

Example 2.9 *Each of the following is an atomic wff of QL.*

1. *The wff* $N(a, b)$ *which might be the symbolization of the sentence 'Saskatoon is north of Regina'.*
2. *The wff* $F(d, l, x)$ *which might be the symbolization of the sentence 'Don Wells flies to London with* x*'.*

Using informal mathematical notation (see remark (2.6)), then all of the following are examples of atomic formulas.

(A)

$$x^2 + 2y < 3. \tag{18}$$

Here the predicate symbol is a two-place predicate symbol who's intended interpretation is the less-than relation informally denoted simply by the customary mathematical notation '<'. We could write (18) more formally; let f_1^2 and A_3^2 stand respectively for the functional operation of addition and the two-place predicate symbol who's intended interpretation is the less-than relation. Moreover, f_2^2 and f_1^1 might respectively be interpreted as multiplication and and the square function. Accordingly, $x^2+2y < 3$ would be written more formally as

$$A_3^2\left(f_1^2\left(f_1^1(x), f_2^2(2,y)\right), 3\right). \tag{19}$$

Here we did not, as we did in example (2.5), make use of the identity function $f(t) = t$.

(B)

$$\sqrt{x^2+1} = x^2 - 3x^3. \tag{20}$$

What is involved here is the two-place predicate of equality. Since $t_1 := \sqrt{x^2+1}$ and $t_2 := x^2 - 3x^3$ are terms (of course written in an informal manner), (20) corresponds to the atomic wff

$$A_1^2(t_1, t_2), \tag{21}$$

where A_1^2 is the formal two-place predicate letter of which the intended interpretation is equality.

(C)

$$\int_1^\infty f(x)dx \geq \sum_{n=2}^{\infty} f(n). \tag{22}$$

Here the predicate is the two-place predicate informally written as '≥'. The terms are $t_1 := \int_1^\infty f(x)dx$ and $t_2 := \sum_{n=2}^\infty f(n)$.

2.3 Well-Formed Formulas of QL

Definition 2.10 (Wffs) *Let $\mathcal{WF}$ denote the set of all well-formed formulas of QL. Well-formed formulas are recursively constructed from the atomic formulas just defined, in the following manner.*

1. *If $\varphi \in \mathcal{AF}$ then $\varphi \in \mathcal{WF}$.*
2. *If $\varphi, \psi \in \mathcal{WF}$ then $(\varphi \otimes \psi) \in \mathcal{WF}$, where $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$.*
3. *If $\varphi \in \mathcal{WF}$ then $\neg\varphi \in \mathcal{WF}$.*
4. *If $\varphi \in \mathcal{WF}$ then $\mathbb{J}v\varphi \in \mathcal{WF}$, where $\mathbb{J} \in \{\forall, \exists\}$ and v represents (is a placeholder for) any individual variable in the set $\mathcal{V}$ of individual variables.*
5. *Nothing belongs to the set $\mathcal{WF}$ of well-formed formulas of QL unless it can be generated by means of (1)–(4).*

Example 2.11 *According to the above definition each wff of QL is constructed from the basic units, the atomic wffs. Consider the following set of atomic wffs:*

$$\{S(x), C(y), F(x, y, a)\}. \tag{23}$$

(i) By (2) of the above definition

$$(C(y) \wedge F(x, y, a)) \tag{24}$$

is a wff of QL.

(ii) *By (4) of the above definition*

$$\exists y(C(y) \wedge F(x,y,a)) \tag{25}$$

is a wff of QL.

(iii) *By (2) of the above definition*

$$(S(x) \Longrightarrow \exists y(C(y) \wedge F(x,y,a))) \tag{26}$$

is a wff of QL.

(iv) *By (4) of the above definition*

$$\forall x(S(x) \Longrightarrow \exists y(C(y) \wedge F(x,y,a))) \tag{27}$$

is a wff of QL

3 Additional Syntactic Structures and Concepts

3.1 Subformulas of QL

Definition 3.1 (Subformula) *Let $SF[\varphi]$ denote the set of all subformulas of φ. Then*

1. *If $\varphi \in \mathcal{AF}$ then $SF[\varphi] \stackrel{def}{=} \{\varphi\}$.*
2. *If $\varphi \in \mathcal{WF}$ is of the form $(\neg\psi)$ then $SF[\varphi] \stackrel{def}{=} \{\varphi\} \cup SF[\psi]$*
3. *If $\varphi \in \mathcal{WF}$ is of the form $(\psi \otimes \chi)$, then $SF[\varphi] \stackrel{def}{=} \{\varphi\} \cup SF[\psi] \cup SF[\chi]$, where $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$.*
4. *If $\varphi \in \mathcal{WF}$ is of the form $\mathrm{Jl} v\psi$, then $SF[\varphi] \stackrel{def}{=} \{\varphi\} \cup SF[\psi]$.*
5. *For any $\varphi \in \mathcal{WF}$, nothing is a subformula of φ unless it can be shown to be so by means of (1)–(4).*

Remark 3.2 *Study the table on page 301 of the Logic Book.*

Definition 3.3 (Proper Subformula) *If $\varphi, \psi \in \mathcal{WF}$, then ψ is a* **proper** *subformula of φ iff $\psi \in SF[\varphi]$ and $\psi \neq \varphi$. Let $\overline{SF}[\varphi]$ denote the set of all proper subformulas of φ.*

Definition 3.4 (Main Logical Operator (MLO)) *The main logical operator of a wff φ is defined as follows:*

1. *If φ is of the form $\neg\psi$ then the initial '$\neg$' is the MLO of φ.*
2. *If φ is of the form $(\psi \otimes \chi)$, where $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$, then '$\otimes$' is the MLO of φ.*
3. *If φ is of the form $\mathrm{Jl} v\varphi$ then $\mathrm{Jl} v$ is the MLO of φ.*
4. *For any $\varphi \in \mathcal{WF}$, nothing is the main logical operator of φ unless it can be shown to be so by means of (1)–(3).*

Definition 3.5 (Quantifier Scope) *The scope of a quantifier in $\varphi \in \mathcal{WF}$ is the subformula ψ of φ of which that quantifier is the main logical operator.*

Remark 3.6 *Attaching a quantifier to $\psi \in \mathcal{WF}$ produces a new wff $\varphi \in \mathcal{WF}$, namely, $\varphi := \mathrm{Jl} v\psi$, of which the quantifier is the main logical operator. The scope of the quantifier is all of the wff ψ to which the quantifier is being attached.*

3.2 Bondage and Freedom of Variables

Definition 3.7 (Bound Occurrence of Variable) *An occurrence of a variable v_i in $\varphi \in \mathcal{WF}$ is bound iff that occurrence either is in a quantifier expression '$\text{Jl}\, v_i$' in φ or lies within the scope of an v_i-quantifier $\text{Jl}\, v_i$ in φ.*

Definition 3.8 (Free Occurrence of Variable) *An occurrence of a variable v_i in $\varphi \in \mathcal{WF}$ is free iff it is not bound.*

Definition 3.9 (Set of free variables of a term) *Let $t \in \mathcal{T}$. The set $FV[t]$ of free variables of t is defined by*

1. $FV[v_i] \overset{def}{=} \{v_i\}$
2. $FV[\mathfrak{a}_i] \overset{def}{=} \emptyset; \quad \mathfrak{a}_i \in \mathcal{C}$.
3. $FV[\mathfrak{f}_i^n(t_1, \ldots, t_n)] \overset{def}{=} FV[t_1] \cup \cdots \cup FV[t_n]$.

Definition 3.10 (Set of free variables of a wff) *Let $\varphi \in \mathcal{WF}$. Then the set $FV[\varphi]$ of free variables of φ is defined as follows.*

1. *If* $\varphi \in \mathcal{AF}$ *then* $FV[\varphi_i^n(t_1, \ldots, t_n)] \overset{def}{=} FV[t_1] \cup, \ldots, \cup FV[t_n]$.
2. $FV[t_i = t_j] \overset{def}{=} FV[t_i] \cup FV[t_j]; \quad i, j \in I \subseteq \mathbb{N}$.
3. $FV[\bot] \overset{def}{=} \emptyset$.
4. $FV[\varphi \otimes \psi] \overset{def}{=} FV[\varphi] \cup FV[\psi]$, *where* $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$.
5. $FV[\neg\varphi] \overset{def}{=} FV[\varphi]$.
6. $FV[\text{Jl}\, v_i \varphi] \overset{def}{=} FV[\varphi] \backslash \{v_i\}$.

Definition 3.11 (closed terms, closed wffs) *$t \in \mathcal{T}$ and $\varphi \in \mathcal{WF}$ are called* **closed** *terms and* **closed** *well-formed formulas respectively iff $FV[t] = \emptyset$ and $FV[\varphi] = \emptyset$ respectively. The set of closed terms and closed formulas will be denoted respectively by $\overline{\mathcal{T}}$ and $\overline{\mathcal{WF}}$.*

Definition 3.12 (Sentence of QL) *A closed well-formed formula is called a sentence (or proposition). $\overline{\mathcal{WF}}$ denotes the set of all sentences of QL.*

Remark 3.13 *Let $BV[\varphi]$ denote the set of all* **bound** *variables of φ. It should be noted that $FV[\varphi] \cap BV[\varphi]$ need not be empty. For example,*

$$\forall x_1(x_1 = x_2) \Longrightarrow P(x_1) \tag{28}$$

contains x_1 both free and bound.

3.3 The Operation of Replacement

We first introduce the operation $[t/v]$ of replacing a variable v with some term t, *for terms*. This operation does not take into account the possibility of new 'quantifier capture' of variables as a result of the operation. However, the subsequent definition of *substitutability* imposes certain restrictions on the operation of replacement with respect to the possible emergence of new relationships between quantifiers and variables.

Definition 3.14 *Let s and t be any terms and let $v, w \in \mathcal{V}$ be some variables. The notation $s[t/v]$ says: All the occurrences of the variable v in the term s are* **replaced** *by the term t. Note that since s and t are any terms, s may be a variable, an individual constant or a function. We define $s[t/v]$ accordingly by*

1. $w[t/v] \overset{def}{=} \begin{cases} w & \text{iff} \quad w \neq v \\ t & \text{iff} \quad w = v; \end{cases}$

2. $\mathfrak{a}[t/v] \overset{def}{=} \mathfrak{a}; \qquad \mathfrak{a} \in \mathcal{C}$

3. $\mathfrak{f}_i^n(t_1, \ldots, t_n)[t/v] \overset{def}{=} \mathfrak{f}_i^n(t_1[t/v], \ldots, t_n[t/v])$.

Remark 3.15 *Note that if $w \neq v$ (first clause of the definition) then v does not occur in w. Consequently, if v does not occur in w it cannot be replaced with some term t. Hence there will be no change and $w[t/v] = w$.*

For example, consider $x[t/y]$. By definition $x[t/y]$ says: replace every occurrence of y in x with the term t. Clearly y does not occur in x; therefore y cannot be replaced by t. Hence, $x[t/y] = x$; that is, there is no change.

If $w = v$ then v **does** *occur in w and $w[t/v]$ says replace v with t, and since $w = v$, this means replace w with t; hence, $w[t/v] = t$.*

For example, consider $x[t/x]$. By definition $x[t/x]$ says: replace every occurrence of x in x with the term t. Clearly, x occurs in x. Therefore, $x[t/x] = t$.

The next definition introduces the replacement operation $[t/v]$ that replaces a variable v with some term t, *for wffs* of QL.

Definition 3.16 *If $\varphi \in \mathcal{WF}$ and $v \in \mathcal{V}$, and $t \in \mathcal{T}$, then $\varphi[t/v]$ is the result of replacing every* **free** *occurrence of v in φ by t. For any $\varphi \in \mathcal{WF}$, $\varphi[t/v]$ is defined by*

1. $\bot[t/v] \overset{def}{=} \bot$

2. $\varphi_i^n(t_1, \ldots, t_n)[t/v] \overset{def}{=} \varphi_i^n(t_1[t/v], \ldots, t_n[t/v])$

3. $(t_i = t_j)[t/v] \overset{def}{=} (t_i[t/v] = t_j[t/v]); \qquad i, j \in I \subseteq \mathbb{N}$

4. $(\varphi \otimes \psi)[t/v] \overset{def}{=} (\varphi[t/v] \otimes \psi[t/v]) \qquad$ *where* $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$

5. $(\neg\varphi)[t/v] \overset{def}{=} \neg\varphi[t/v]$

6. $(\rrbracket w\varphi)[t/v] \overset{def}{=} \begin{cases} \rrbracket w\varphi[t/v] & \text{iff} \quad w \neq v \\ \rrbracket v\varphi & \text{iff} \quad v = w \end{cases}$

Remark 3.17 *Items (1) and (6) may require some explanation. Let us first consider item (1). Recall that $\bot$ is an atomic wff with no free variables, that is, $FV[\bot] = \emptyset$. Consequently, no free variable occurs in the* *atomic wff $\perp$ of QL which can be replaced with the term t. As a result there will be no change. In fact the following is true in general.*

$$\text{Any term } t \text{ can be used to replace } v \text{ in a wff } \varphi, \text{ if } \varphi \text{ contains no free occurrences of } v. \tag{29}$$

To see this consider definition (3.16) again. According to this definition, the replacement of v in φ with t, that is, $\varphi[t/v]$, is the result of replacing every **free** *occurrence of v in φ with the term t. Expressed more formally the definition says:*

For any variable v, for any wff and for any term t, if $v \in FV[\varphi]$ then $\varphi[t/v]$.

This is a statement which occurs in the meta-language; using meta-quantifiers and meta-connectives, it can be written as

$$\forall\!\!\forall v \forall\!\!\forall \varphi \forall\!\!\forall t (v \in FV[\varphi] \Longrightarrow \varphi[t/v]). \tag{30}$$

Clearly this statement is true if for all v, $v \notin FV[\varphi]$, that is, the statement is true if φ contains no free occurrences of v, or if $\forall v(v \in FV[\varphi])$ is false and $\forall v(v \notin FV[\varphi])$ is true. Recall that a conditional with false antecedent is always true. Clearly, 29 is a special case covered by definition (3.16), namely, the vacuous *case.*

The case of vacuous replacement: *If $v \notin FV[\varphi]$ then $\varphi[t/v] = \varphi$, that is, no change occurs.*

The case of non-vacuous replacement: *If $v \in FV[\varphi]$ and if $t \neq v$ then $\varphi[t/v] \neq \varphi$, that is, change does occur.*

As an example of the vacuous case consider the following wff and term:

$$\varphi := \forall x_1 A_1^2(x_1, x_2) \tag{31}$$

$$t := f_1^2(x_1, x_2). \tag{32}$$

Can the term $t := f_1^2(x_1, x_2)$ replace x_1 in $\varphi := \forall x_1 A_1^2(x_1, x_2)$? The answer is yes because $x_1 \notin FV[\varphi]$.

$$\varphi[t/x_1] = \forall x_1 A_1^2(x_1, x_2) \left[f_1^2(x_1, x_2)/x_1\right] \tag{33}$$

$$= \forall x_1 A_1^2(x_1, x_2) \tag{34}$$

$$= \varphi \tag{35}$$

That is, in the vacuous case no change occurs, and $\varphi[t/v] = \varphi$.

Consider item (6) in definition (3.16). Suppose $v = w$, and let $\psi := \text{Jl} w\varphi = \text{Jl} v\varphi$. Since $w \notin FV[\psi]$ and $v \notin FV[\psi]$, $\psi[t/v] = \psi = \text{Jl} v\varphi$ for any term t of QL.

Suppose $w \neq v$, and let $\psi := \text{Jl} w\varphi$. Then $FV[\psi] = FV[\varphi] \backslash \{w\}$. Therefore, either

1. *$v \in FV[\varphi] \backslash \{w\}$, or*
2. *$v \notin FV[\varphi] \backslash \{w\}$.*

If the first alternative is the case then $v \in FV[\varphi]$ and we have the non-vacuous situation. Therefore $(\text{Jl} w\varphi)[t/v] = \text{Jl} w\varphi[t/v]$ and $\varphi[t/v] \neq \varphi$ provided $t \neq v$. If the second alternative is the case, then $v \notin FV[\varphi]$, and we have the vacuous situation. Therefore, $(\text{Jl} w\varphi)[t/v] = \text{Jl} w\varphi[t/v] = \text{Jl} w\varphi$.

Consider the above wff and term again, namely,

$$\varphi := \forall x_1 A_1^2(x_1, x_2) \tag{36}$$

$$t := f_1^2(x_1, x_2). \tag{37}$$

Can the term $t := f_1^2(x_1, x_2)$ replace x_2 in $\varphi := \forall x_1 A_1^2(x_1, x_2)$? The answer is yes because $x_2 \in FV[\varphi] \backslash \{x_1\}$. The condition of replacement is satisfied non-vacuously.

$$\varphi[t/x_1] = \forall x_1 A_1^2(x_1, x_2) \left[f_1^2(x_1, x_2)/x_2\right] \tag{38}$$

$$= \forall x_1 A_1^2(x_1, f_1^2(x_1, x_2)) \tag{39}$$

The next definitions make precise what it means for a term to be **substitutable** for a variable v in a well-formed formula φ. That is, the concept of 'substitutability' imposes certain restrictions on the operation of replacement introduced above.

Definition 3.18 *A variable w is* **substitutable** *for a variable v in φ iff $\varphi[w/v]$ and no* **free** *occurrence of v in φ becomes a* **bound** *occurrence of w in $\varphi[w/v]$.*

Remark 3.19 *One also sometimes expresses the above concept in a different way. One says for example: A wff φ is said to* **admit** *a variable w for a variable v of φ iff w is not bound by a quantifier in φ whenever w is substituted for any specific free occurrence of v in φ. Put differently: A wff φ* **admits** *a variable w for a variable v in φ iff no* **free** *occurrence of v in φ becomes a* **bound** *occurrence of w in $\varphi[w/v]$, that is, if and only if w is substitutable for v in φ.*

Definition 3.20 *A term t is* **substitutable** *for a variable v in φ iff $\varphi[t/v]$ and every variable of t is* **substitutable** *for v in φ.*

Remark 3.21 *Put differently, a term t is substitutable for v_i in φ iff no free occurrence of v_i in φ lies within the scope of a quantifier $\text{Л}\| v_j$, and v_j is a variable occurring in t. Roughly, this means that t may be substituted for every free occurrence of v_i in φ provided that no new interactions with quantifiers in φ are introduced.*

Remark 3.22 *Consider the above wff and term yet again, namely,*

$$\varphi := \forall x_1 A_1^2(x_1, x_2) \tag{40}$$

$$t := f_1^2(x_1, x_2). \tag{41}$$

We saw that the term $t := f_1^2(x_1, x_2)$ can replace *x_2 in $\varphi := \forall x_1 A_1^2(x_1, x_2)$. But is the term $t := f_1^2(x_1, x_2)$ substitutable for x_2 in $\forall x_1 A_1^2(x_1, x_2)$? The answer is no, since it is not the case that every variable of $t := f_1^2(x_1, x_2)$ is substitutable for x_2 in $\forall x_1 A_1^2(x_1, x_2)$. The variable x_1 of the term $t := f_1^2(x_1 x_2)$ is not substitutable for the free occurrence of x_2 in $\forall x_1 A_1^2(x_1, x_2)$ since such a free occurrence of x_2 in $\forall x_1 A_1^2(x_1, x_2)$ becomes a bound occurrence of x_1 in $\forall x_1 A_1^2(x_1, x_2)[x_1/x_2]$.*

Example 3.23 *Consider the wff*

$$\forall x_1 A_1^2(x_1, x_2) \Longrightarrow \forall x_3 A_2^2(x_3, x_1). \tag{42}$$

Here, for example, $f_1^2(x_1, x_4)$ is not substitutable for x_2; $f_2^2(x_2, x_3)$ is substitutable for x_2; x_2 is substitutable for x_1 (note that x_1 occurs freely only once); and $f_4^2(x_1, x_3)$ is not substitutable for x_1 but $f_1^2(x_1, x_4)$ is substitutable for x_1.

The next definition makes precise the notion of 'substitutable' for wffs of various complexity.

Definition 3.24 *The term t is* **substitutable** *for a variable v in φ iff*

1. *φ is an atomic wff, that is, $\varphi \in \mathcal{AF}$.*

2. *$\varphi \stackrel{def}{=} (\varphi_1 \otimes \varphi_2)$ and t is substitutable for v in φ_1 and φ_2, where $\otimes \in \{\wedge, \vee, \Longrightarrow, \Longleftrightarrow\}$.*

3. *$\varphi \stackrel{def}{=} \neg\varphi_1$ and t is substitutable for v in φ_1.*

4. *$\varphi \stackrel{def}{=} \text{Л}\| w\psi$ and $w \notin FV[t]$ and t is substitutable for v in ψ.*

4 Examples

Example 4.1

$$P_1^2(x_1, x_2) \tag{43}$$

$$P_1^2(x_1, x_2) \Longrightarrow \forall x_1 P_1^1(x_1) \tag{44}$$

$$\forall x_1 (P_1^2(x_1, x_2) \Longrightarrow \forall x_1 P_1^1(x_1)) \tag{45}$$

$$\exists x_1 P_1^2(x_1, x_2) \tag{46}$$

In (43) the single occurrence of x_1 is free. In (44) the first occurrence of x_1 is free, but the second and third occurrences are bound. In (45) all occurrences of x_1 are bound. And in (46) both occurrences of x_1 are bound.

In all four wffs, every occurrence of x_2 is free. Notice that (as in (44)) a variable may have both free and bound occurrences in the same wff. Also observe that an occurrence of a variable may be bound in some wff φ but free in a subformula of φ. For example, the first occurrence of x_1 is free in (44) but bound in (45).

Example 4.2 *The following should be reasonably clear.*

1. *The term x_2 is substitutable for x_1 in $A_1^1(x_1)$, but x_2 is not substitutable for x_1 in $\forall x_2 A_1^1(x_1)$.*
2. *The term $f_1^2(x_1, x_3)$ is substitutable for x_1 in $\forall x_2 A_1^2(x_1, x_2) \Longrightarrow A_1^1(x_1)$ but is not substitutable for x_1 in $\exists x_3 \forall x_2 A_1^2((x_1, x_2) \Longrightarrow A_1^1(x_1))$.*

Example 4.3 *The following facts are obvious:*

1. *A term that contains no variables is substitutable for any free variable in any wff.*
2. *A term t is substitutable for any variable in φ if none of the variables of t is bound in φ.*
3. *x_i is substitutable for x_i in any wff.*

Example 4.4 *The following should be reasonably clear.*

1. *x_2 is substitutable for x_0 in the wff $\exists x_3 A^2(x_0, x_3)$.*
2. *The term $f^2(x_0, x_1)$ is not substitutable for x_0 in the wff $\exists x_1 A^2(x_0, x_3)$.*
3. *x_5 is substitutable for x_1 in $A^2(x_1, x_3) \Longrightarrow \exists x_1 Q^2(x_1, x_2)$.*

If we wish our first order language to be appropriate for statements about the arithmetic of natural numbers, then we might take our language to have (besides, variables, punctuation, connectives and quantifiers) the symbols:

a_1 to stand for the natural number 0;

A_1^2 to stand for $=$

f_1^1 to stand for the successor function;

f_1^2 to stand for $+$;

f_2^2 to stand for $\times$.

Then

$$A_1^2(f_1^2(x_1, x_2), f_2^2(x_1, x_2)) \tag{47}$$

would be interpreted as

$$x_1 + x_2 = x_1 x_2. \tag{48}$$

If we wish our first order language to be appropriate for statements about groups, then we might take our language to have (besides, variables, punctuation, connectives and quantifiers) the symbols:

a_1 to stand for the identity element.

A_1^2 to stand for '='.

f_1^1 to stand for the function which takes each group element to its inverse.

f_1^2 to stand for the binary group operation '$\circ$'.

For example,

$$A_1^2(f_1^2(x_1, f_1^1(x_1)), a_1) \tag{49}$$

would be interpreted as

$$x_1 \circ x_1^{-1} = \text{identity}. \tag{50}$$

Example 4.5 *Consider the following symbols:*

Predicate symbols: $L, =$, *where* L *is interpreted as 'less than'.*

Function symbols: f_1^2, f_2^1 *which function respectively as 'product' and 'inverse'.*

Constant symbol: e *which is the identity.*

Some terms:

$$t_1 := x_0 \tag{51}$$
$$t_2 := f_1^2(x_1, x_2) \tag{52}$$
$$t_3 := f_1^2(e, e) \tag{53}$$
$$t_4 := f_2^1(x_7) \tag{54}$$
$$t_5 := f_1^2(f_2^1(f_1^2(x_2, e)), f_2^1(x_1)). \tag{55}$$

Some formulas:

$$\varphi_1 := x_0 = x_2 \tag{56}$$
$$\varphi_2 := t_3 = t_4 \tag{57}$$
$$\varphi_3 := L(f_2^1(x_5), e) \tag{58}$$
$$\varphi_4 := ((x_0 = x_1) \Longrightarrow (x_1 = x_0)) \tag{59}$$
$$\varphi_5 := \forall x_0 \forall x_1 ((x_0 = x_1) \Longrightarrow \neg L(x_0, x_1)) \tag{60}$$
$$\varphi_6 := \forall x_0 \exists x_1 (f_1^2(x_0, x_1) = e) \tag{61}$$
$$\varphi_7 := \exists x_1 (\neg(x_1 = e) \wedge f_1^2(x_1, x_1) = e) \tag{62}$$

Then we have the following:

$$FV[t_2] = \{x_1, x_2\} \tag{63}$$
$$FV[t_3] = \emptyset \tag{64}$$
$$FV[\varphi_2] = FV[t_3] \cup FV[t_4] = \{x_7\} \tag{65}$$
$$FV[\varphi_7] = \emptyset \tag{66}$$
$$BV[\varphi_4] = \emptyset \tag{67}$$
$$BV[\varphi_6] = \{x_0, x_1\} \tag{68}$$
$$t_4[t_2/x_1] = f_2^1(x_7) \tag{69}$$
$$t_4[t_2/x_7] = f_2^1(f_1^2(x_1, x_2)) \tag{70}$$
$$t_5[x_2/x_1] = f_1^2(f_2^1(f_1^2(x_2, e)), f_2^1(x_2)) \tag{71}$$
$$\varphi_1[t_3/x_0] = f_1^2(e, e) = x_2 \tag{72}$$
$$\varphi_5[t_3/x_0] = \varphi_5 \tag{73}$$

5 Exercises

Classify each of the following expressions as

(i) terms,

(ii) atomic wffs,

(iii) wffs,

(iv) sentences, that is, closed wffs,

(v) none of these.

(Some expressions may be in more than one classification.)

(a) Oscar and Miriam.

(b) x is older than y.

(c) The shortest person in x's class.

(d) Oscar dates every girl in his class.

(e) $\sqrt{x^2 - (y^3 + 7)}$

(f) $3 > 1 + 8$

(g) $(x \wedge y) > 0$

(h) $x < (y < z)$

(i) $\forall x(x - x = 0)$

6 Solutions to exercises

(a) "Oscar" and "Miriam" are terms. We cannot connect terms with sentential connectives in our language. Therefore, (a) is neither a term nor a formula, so the answer is (v). (In order to translate into the Predicate Language the sentence "Oscar and Miriam are students," we would rewrite it as "Oscar is a student and Miriam is a student.")

(b) This is an atomic wff. It would be translated as $O(x, y)$. Because every atomic wff is a wff of QL, (ii) and (iii) is the correct answer.

(c) This expression is a term, so (i) is correct.

(d) This is a sentence, that is, a closed wff. We could translate it into our Predicate Language as $\forall x(G(x) \Longrightarrow D(o, x))$. Because every closed wff is a wff, (iii) and (iv) is the answer.

(e) This is a term; hence (i).

(f) The expression $\ulcorner 3 > 1 + 8 \urcorner$ is an atomic wff. The predicate is the 'greater than' relation informally expressed as $\ulcorner > \urcorner$, and the terms are 3 and $1 + 8$. It is also a sentence, that is, a closed wff of QL, because there are no free variables. (It happens to be false, but that is irrelevant for our purposes here.) Thus, (ii), (iii) and (iv) is the answer.

(g) This expression is meaningless in our language. A sentential connective cannot connect two terms: therefore, as written, the answer is (v). When students write $\ulcorner (x \wedge y) > 0 \urcorner$ they usually mean $\ulcorner (x > 0) \wedge (y > 0) \urcorner$. The latter expression is the way it has to be written in QL.

(h) As written, expression (h) is meaningless. The predicate $<$ is a two-place predicate. The expression $\ulcorner y < z \urcorner$ is a wff, not a term. It is meaningless to say that a term is less than a wff. A term can only be less than another term. Thus (v) is the correct answer. Frequently, in mathematics, we write $\ulcorner x < y < z \urcorner$ as an abbreviation for $\ulcorner (x < y) \wedge (y < z) \urcorner$. The latter expression is a wff.

(i) This is a wff with no free variables. It is not atomic because it contains a quantifier; thus, it is a sentence, a closed wff. The answer is (iii) and (iv).

7 More exercises

Exercise 7.1 *Classify each of the following expressions as*

i terms,

ii atomic wffs,

iii wffs that are not atomic,

iv sentences (closed wffs)

v none of these.

Note that some expressions may be in more than one classification. **Provide reasons for your answers**.

1. *The tallest student in the class.*
2. *Pink flowers.*
3. *Nancy types rapidly and accurately.*
4. *Accurately.*

5. $1 \leq \sqrt{3}$
6. $(x + y = -(x + y))^2$
7. $(x \vee y) < z$
8. $\forall x((x < P(x)) \vee (P(x) < x))$
9. $\forall x(\exists y(x + y = z) \Longrightarrow (y = 0))$

Exercise 7.2 *Let $t := f_1^2(x_1, x_2)$. Determine $\varphi[t/x_1]$ for the following wffs of QL.*

1. $\varphi := \forall x_2 A_1^2(x_2, f_1^2(x_1, x_2)) \Longrightarrow A_1^1(x_1)$.
2. $\varphi := \forall x_1 \forall x_3 (A_1^1(x_3) \Longrightarrow A_1^1(x_1))$.
3. $\varphi := \forall x_2 A_1^1(f_1^1(x_2)) \Longrightarrow \forall x_3 A_1^3(x_1, x_2, x_3)$.
4. $\varphi := \forall x_2 A_1^3(x_1, f_1^1(x_1), x_2) \Longrightarrow \forall x_3 A_1^1(f_1^2(x_1, x_3))$.

Exercise 7.3 *Let $t := f_1^2(x_1, x_2)$. Determine whether t is substitutable for x_1 in φ for the following wffs of QL.*

1. $\varphi := \forall x_2 A_1^2(x_2, f_1^2(x_1, x_2)) \Longrightarrow A_1^1(x_1)$.
2. $\varphi := \forall x_1 \forall x_3 (A_1^1(x_3) \Longrightarrow A_1^1(x_1))$.
3. $\varphi := \forall x_2 A_1^1(f_1^1(x_2)) \Longrightarrow \forall x_3 A_1^3(x_1, x_2, x_3)$.
4. $\varphi := \forall x_2 A_1^3(x_1, f_1^1(x_1), x_2) \Longrightarrow \forall x_3 A_1^1(f_1^2(x_1, x_3))$.

Exercise 7.4 *Let $t := f_1^2(x_1, x_3)$. Determine $\varphi[t/x_1]$ for the following wffs of QL.*

1. $\varphi := \forall x_2 A_1^2(x_2, f_1^2(x_1, x_2)) \Longrightarrow A_1^1(x_1)$.
2. $\varphi := \forall x_1 \forall x_3 (A_1^1(x_3) \Longrightarrow A_1^1(x_1))$.
3. $\varphi := \forall x_2 A_1^1(f_1^1(x_2)) \Longrightarrow \forall x_3 A_1^3(x_1, x_2, x_3)$.
4. $\varphi := \forall x_2 A_1^3(x_1, f_1^1(x_1), x_2) \Longrightarrow \forall x_3 A_1^1(f_1^2(x_1, x_3))$.

Exercise 7.5 *Let $t := f_1^2(x_1, x_3)$. Determine whether t is substitutable for x_1 in φ for the following wffs of QL.*

1. $\varphi := \forall x_2 A_1^2(x_2, f_1^2(x_1, x_2)) \Longrightarrow A_1^1(x_1)$.
2. $\varphi := \forall x_1 \forall x_3 (A_1^1(x_3) \Longrightarrow A_1^1(x_1))$.
3. $\varphi := \forall x_2 A_1^1(f_1^1(x_2)) \Longrightarrow \forall x_3 A_1^3(x_1, x_2, x_3)$.
4. $\varphi := \forall x_2 A_1^3(x_1, f_1^1(x_1), x_2) \Longrightarrow \forall x_3 A_1^1(f_1^2(x_1, x_3))$.