Philosophy 352

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Contents

1 Numerical Quantification

Consider the following numerical claims:

1. *At least two* ipads arrived today.
2. *At most two* ipads are missing from the office.
3. *Exactly two* ipads are on the table.

It is possible in quantification logic to express these three kinds of numerical claims.

1.1 At least n objects

Recall that in the language of quantification logic each name refers to one and only one object; however, distinct names need not necessarily refer to distinct objects. Similarly, distinct variables need not vary over distinct objects. For example, both of the following sentences can be made true in a world or domain with one object. Let $C(u)$ stand for "u is a cat", and $S(u)$ stand for "u is small".

$$C(a) \wedge S(a) \wedge C(b) \tag{1}$$

$$\exists x \exists y [C(x) \wedge S(x) \wedge C(y)] \tag{2}$$

In order to be clear that there are at least two cats, you must find a way of showing that there really are two different cats. You could do this in the following two ways:

$$C(a) \wedge S(a) \wedge C(b) \wedge L(b) \tag{3}$$

$$\exists x \exists y [C(x) \wedge S(x) \wedge C(y) \wedge R(x, y)] \tag{4}$$

where $L(u)$ stands for "u is large", and $R(x, y)$ stands for "x is to the right of y". Of course, the most direct way of making it clear that there really are two cats is as follows:

$$\exists x \exists y [C(x) \wedge S(x) \wedge C(y) \wedge (x \neq y)] \tag{5}$$

There are at least three cats could then be expressed as

$$\exists x \exists y \exists z[(C(x) \wedge C(y) \wedge C(z)) \wedge ((x \neq y) \wedge (x \neq z) \wedge (y \neq z))] \tag{6}$$

To say that there are at least four cats requires four existential quantifiers and six inequalities:

$$x \neq y, x \neq z, x \neq z_1, y \neq z, y \neq z_1, z \neq z_1.$$

1.2 At most n objects

How can we say that there are *at most* two cats? Well one way of doing this is by saying: "It is not the case that there are *at least* three cats", which is the negation of (6), namely,

$$\neg \exists x \exists y \exists z[(C(x) \wedge C(y) \wedge C(z)) \wedge ((x \neq y) \wedge (x \neq z) \wedge (y \neq z))]. \tag{7}$$

But (7) is equivalent to (check this out yourself!)

$$\forall x \forall y \forall z[(C(x) \wedge C(y) \wedge C(z)) \Longrightarrow ((x = y) \vee (x = z) \vee (y = z))] \tag{8}$$

Notice, it takes two existential quantifiers to express *there are at least two cats*; but it takes three universal quantifiers to say *there are at most two cats.* More generally, to translate "there are at least n objects" takes n existential quantifiers; but to say "there are at most n objects" takes $n + 1$ universal quantifiers.

1.3 Exactly n objects

The expression

(a) there are exactly two cats

means the conjunction of the following two expressions:

(b) there are at least two cats
(c) there are at most two cats.

Thus (a) means

$$\begin{aligned} &\exists x \exists y[C(x) \wedge C(y) \wedge (x \neq y)] \\ &\qquad \wedge \forall x \forall y \forall z[(C(x) \wedge C(y) \wedge C(z)) \Longrightarrow ((x = y) \vee (x = z) \vee (y = z))] \end{aligned} \tag{9}$$

The left conjunct symbolizes (b) and the right conjunct symbolizes (c).

However, claim (a) "there are exactly two cats" (in some universe of discourse, that is, some specified domain), can be symbolized much more parsimoniously as follows:

$$\exists x \exists y\{[C(x) \wedge C(y) \wedge (x \neq y)] \wedge \forall z[C(z) \Longrightarrow ((z = x) \vee (z = y))]\} \tag{10}$$

If we translate formula (10) into English, we observe that the formula says: "There are two distinct objects, both of which are cats, and consider anything in the universe of discourse, if it is a cat, then it is one of these two cats." As an exercise, you might want to show that the wffs (9) and (10) are equivalent. Another way of symbolizing "there are exactly two cats" is

$$\exists x \exists y \forall z[(x \neq y) \wedge (C(z) \Longleftrightarrow ((z = x) \vee (z = y)))]. \tag{11}$$

We can abbreviate numerical quantificational claims as follows:

1. $\exists^{\geq n}\varphi(x)$ stands for the wff asserting "There are at least n objects satisfying $\varphi(x)$.

2. $\exists^{\leq n}\varphi(x)$ stands for the wff asserting "There are at most n objects satisfying $\varphi(x)$.

3. $\exists^{!\,n}\varphi(x)$ stands for the wff asserting "There are exactly n objects satisfying $\varphi(x)$.

The claim that there is exactly one object satisfying $\varphi(x)$ is simply abbreviated as $\exists^{!}\varphi(x)$, instead of $\exists^{!1}x\varphi(x)$; that is,

$$\exists^{!}x\varphi(x) \stackrel{\text{def}}{=} \exists x[\varphi(x) \wedge \forall y(\varphi(y) \Longrightarrow (y = x))]. \tag{12}$$

It is very important here to make sure that y is substitutable in the wff $\varphi(x)$. Suppose $\varphi(x) := \forall y(x < y)$. Then clearly substituting for the **free** variable x with a variable that becomes **bound** to a quantifier in $\varphi(x)$ means that such a substitution instance would change the quantificational relationship! And this is exactly what would happen if you were to substitute for x with y: $\varphi(y/x) = \forall y(y < y)$. $FV[\varphi(x)] = \{x\}$, but after the substitution $FV[\varphi(y/x)] = \emptyset$.