Philosophy 352

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Contents

1 Polyadic Quantification Logic Involving Multiple General Propositions and Identity

1.1 Derivation Rules of SD^{+}

1.2 Quantification Derivation Rules

Remark 1.1 *It should be noted that the Derivation Rules of QL presented below do not permit any open wffs to occur anywhere in the derivations. A variable is either bound locally, that is, the variable in question occurs within the scope of a quantifier* ℿ *of the wff in which it occurs, or the variable in question is bound globally, that is, the variable is in the scope of some* commonizing quantifier ℿ*. Recall that* ℿ $\in \{\forall, \exists\}$.

Remark 1.2 *We will use* u, v, w*, with or without subscripts, as meta-variables which range over the set of variable* $\{x, y, z; x_1, x_2 \ldots\}$*. We will use* $\{r, s, t, \}$*, with or without subscripts, as meta-variables which range over the terms.*

We will use the metavariable $\{\mathfrak{a}\}$*, with or without subscripts, to range over the constant symbols of QL.*

1.2.1 Existential Generalization (EG)

$$\begin{array}{l|ll} & \vdots & \\ \text{i} & \varphi(t) & \\ & \vdots & \\ \text{j} & \exists v \varphi(v) & \text{EG i} \\ & \vdots & \end{array}$$

Instances of (EG)

$$\begin{array}{l|ll} & \vdots & \\ \text{i} & \varphi(a) & \\ & \vdots & \\ \text{j} & \exists v \varphi(v) & \text{EG i} \\ & \vdots & \end{array} \qquad \begin{array}{l|ll} & \vdots & \\ \text{i} & \varphi(u) & \\ & \vdots & \\ \text{j} & \exists v \varphi(v) & \text{EG i} \\ & \vdots & \end{array} \qquad \begin{array}{l|ll} & \vdots & \\ \text{i} & \varphi(u) & \\ & \vdots & \\ \text{j} & \exists u \varphi(u) & \text{EG i} \\ & \vdots & \end{array}$$

Existential Generalization Condition:

1. 't' is a term; therefore, 't' may denote
   (a) an individual constant, or
   (b) an individual variable.
2. Of course, if t denotes an individual variable then that variable must be in the scope of some commonizing quantifier ℿ, where ℿ $\in \{\forall, \exists, \}$.
3. **Principal Condition:** The variable v of the existential generalization (line (j)) must be **new** to the derivation and one may existentially generalize on *any* occurrance of t or on any subset of the occurrances of t.

   This principal condition, which requires, among other things, that the variable v of existential generalization to be new in the derivation, is safe and fairly straightforward. However, it is not strictly necessary that the variable v of existential generalization be new to the derivation, provided the following considerations and restrictions are carefully observed and adhered to:

4. $\varphi(t)$ may itself have quantifiers occurring in it. Therefore, the variable v of existential generalization must be *substitutable* for t in $\varphi(t)$.

5. If $t \neq v$ then v must not occur locally freely elswhere in $\varphi(t)$.

6. If $t = v$, then one must existentially generalize with v on all instances of t in φ and v must not occur elsewhere locally freely in $\varphi(t)$.

Remark 1.3 *When in doubt, existentially generalize on a variable which has not yet occurred in the derivation.*

1.2.2 Universal Instantiation (UI)

$$
\begin{array}{r|ll}
 & \vdots & \\
\text{i} & \forall v \varphi(v, v_1, \ldots, v_k) & \\
 & \vdots & \\
\text{j} & \varphi(t, v_1, \ldots, v_k) & \qquad \text{UI j} \\
 & \vdots &
\end{array}
$$

Universal Instantiation Conditions:

1. 't' is a term; therefore, 't' may denote
   (a) an individual constant, or
   (b) an individual variable,

   because there are no function terms.

2. If t is an individual variable u, then u must be substitutable for v in $\varphi(v, v_1, \ldots, v_k)$.

3. If t is an individual variable u then the u in $\varphi(u, v_1, \ldots, v_k)$ must be in the scope of some commonizing quantifier Л|u and one should make a reference to the line of this commonizing quantifier when one is thus universally instantiating for v with u.

1.2.3 Universal Commonization (UC)

$$
\begin{array}{r|l}
 & \vdots \\
\text{i} & \forall v_1 \varphi_1(v_1) \\
 & \vdots \\
\text{k} & \forall v_k \varphi_k(v_k) \\
 & \vdots \\
 & \begin{array}{|ll}
 \forall v & \qquad \text{UC} \\
 \hline
 \varphi_1(v_1) & \qquad \text{UC i} \\
 \vdots & \\
 \varphi_k(v_k) & \qquad \text{UC k} \\
 \vdots &
 \end{array} \\
 & \vdots
\end{array}
$$

Universal Commonization Conditions:

1. v must not occur locally freely in any $\forall v_i \varphi_i(v_i)$.
2. v is substitutable for v_i in each $\varphi_i(v_i)$.
3. If $v_1 := \cdots := v_k := v$ then conditions (1) and (2) are automatically satisfied. Hence one may universally commonize on v.

$$
\begin{array}{r|l}
 & \vdots \\
\text{i} & \forall v \varphi_1(v) \\
 & \vdots \\
\text{k} & \forall v \varphi_k(v) \\
 & \vdots \\
 & \begin{array}{|ll}
 \forall v & \qquad\qquad \text{UC} \\
 \hline
 \varphi_1(v) & \qquad\qquad \text{UC i} \\
 \vdots & \\
 \varphi_k(v) & \qquad\qquad \text{UC k} \\
 \vdots &
 \end{array} \\
 & \vdots
\end{array}
$$

4. There need not be any $\forall v_i \varphi_i(v_i)$; that is, one may have $k = 0$. Hence

$$
\begin{array}{r|l}
 & \vdots \\
 & \begin{array}{|ll}
 \forall v & \qquad\qquad \text{UC} \\
 \hline
 \vdots &
 \end{array} \\
 & \vdots
\end{array}
$$

1.2.4 Existential Commonization (EC)

$$
\begin{array}{l|l}
 & \vdots \\
\text{i} & \forall v_1 \varphi_1(v_1) \\
 & \vdots \\
\text{j} & \forall v_k \varphi_1(v_k) \\
 & \vdots \\
\ell & \exists v \varphi(v) \\
 & \vdots \\
 & \quad \begin{array}{|ll}
 \exists u & \text{EC} \\
 \hline
 \varphi_1(u) & \text{EC i} \\
 \vdots & \\
 \varphi_k(u) & \text{EC j} \\
 \varphi(u) & \\
 \vdots &
 \end{array} \\
 & \vdots
\end{array}
$$

Existential Commonization Conditions:

1. u must not occur locally freely in $\forall v_i \varphi_i(v_i)$ and $\exists v \varphi(v)$.
2. u is substitutable for v_i in each $\varphi_i(v_i)$ and substitutable for v in $\varphi(v)$.
3. If $v_1 := \cdots := v_k := v$ then conditions (1) and (2) are automatically satisfied. Hence one may existentially commonize on v.

$$
\begin{array}{l|l}
 & \vdots \\
\text{i} & \forall v \varphi_1(v) \\
 & \vdots \\
\text{j} & \forall v \varphi_1(v) \\
 & \vdots \\
\ell & \exists v \varphi(v) \\
 & \vdots \\
 & \quad \begin{array}{|ll}
 \exists v & \text{EC} \\
 \hline
 \varphi_1(v) & \text{EC i} \\
 \vdots & \\
 \varphi_k(v) & \text{EC j} \\
 \varphi(v) & \\
 \vdots &
 \end{array} \\
 & \vdots
\end{array}
$$

4. There need not be any $\varphi_i(v_i)$ or $\varphi(v)$, that is, one may have $k = 0$.

$$
\begin{array}{l|l l}
 & \vdots & \\
 & \quad \exists v & \text{UC} \\
 & \quad \vdots & \\
 & \vdots &
\end{array}
$$

1.2.5 Universal Decommonization (UD) & Existential Decommonization (ED)

$$
\begin{array}{l|l l}
\text{i} & \quad \forall v & \text{UC} \\
 & \quad \vdots & \\
\text{j} & \quad \varphi(v) & \\
\text{k} & \forall v \varphi(v) & \text{UD i–j} \\
 & \vdots &
\end{array}
\qquad
\begin{array}{l|l l}
\text{i} & \quad \exists v & \text{EC} \\
 & \quad \vdots & \\
\text{j} & \quad \varphi(v) & \\
\text{k} & \exists v \varphi(v) & \text{ED i–j} \\
 & \vdots &
\end{array}
$$

Universal-Existential Decommonization Condition:

1. v need not in fact occur in φ. Note, if v does not occur we have vacuous de-quantification. (See vacuous de-quantification rules below.)

2. More than one wff may be decommonized.

1.2.6 Universal Vacuous Quantification (UVQ) & Existential Vacuous Quantification (EVQ)

$$
\begin{array}{l|l l}
 & \vdots & \\
\text{i} & \varphi & \\
 & \vdots & \\
\text{j} & \forall v \varphi & \text{UVQ i} \\
 & \vdots &
\end{array}
\qquad
\begin{array}{l|l l}
 & \vdots & \\
\text{i} & \varphi & \\
 & \vdots & \\
\text{j} & \exists v \varphi & \text{EVQ i} \\
 & \vdots &
\end{array}
$$

Universal & Existential Vacuous Quantification Condition: v does not occur locally freely in φ, that is, $v \notin FV[\varphi]$.

1.2.7 Universal Vacuous Dequantification (UVDQ) & Existential Vacuous Dquantification (EVDQ)

$$
\begin{array}{l|l l}
 & \vdots & \\
\text{i} & \forall v \varphi & \\
 & \vdots & \\
\text{j} & \varphi & \text{UVDQ} \\
 & \vdots &
\end{array}
\qquad
\begin{array}{l|l l}
 & \vdots & \\
\text{i} & \exists v \varphi & \\
 & \vdots & \\
\text{j} & \varphi & \text{EVDQ} \\
 & \vdots &
\end{array}
$$

Universal & Existential Vacuous Dequantification Condition: v does not occur locally freely in φ, that is, $v \notin FV[\varphi]$. Indeed, if v did occur locally freely in φ, then the quantification would not be vacuous.

1.2.8 Reiteration (R)

```
  | ⋮
i | φ
  | ⋮
j | φ            R i
  | ⋮
```

Reiteration Condition: A wff φ that occurs on a given line (i) to the immediate right of a vertical scope line (vertical barrier) of nth order, may be entered on a subsequent line $j > i$ provided it is immediately to the right of the *same unbroken* nth order vertical scope line. This rule is mainly used to make proofs clearer.

1.2.9 Importation (I)

```
  | ← nth vertical scope line (vertical barrier)
  | ⋮
i | φ
  | ⋮
  | ⋱
  | | ← pth vertical scope line (vertical barrier)
  | | ⋮
  | | ⋱
  | | | ← mth vertical scope line (vertical barrier)
  | | | ⋮
  | | |________________________________
  | | | ⋮
j | | | φ                                 I i
  | | | ⋮
  | | ⋮
  | ⋮
```

Importation Condition: A wff φ that occurs on a horizontal line (i) to the right of a vertical scope line (vertical barrier) of order n may be entered on a subsequent line $j > i$ immediately to the right of a vertical scope line of order $m > n$ to the right of the scope line of order n, provided that no vertical scope line of order p, such that $m \geq p > n$, is crossed which is due to the quantifier commonization for a variable that occurs freely (locally) in φ.

1.2.10 Identity Introduction (= I)

Let t be any term.

```
  | ⋮
i | t = t            = I, i
  | ⋮
```

Identity Introduction Conditions:

1. The term t is the variable v say. Then v must be in the scope of a commonizing quantifier Л‖v.
2. Suppose one wishes to introduce identity by means of a term t which is itself a variable v that is not in the scope of some commonizing quantifier Л‖. Then, one must first introduce the required commonizing quantifiers and then introduce identity in order to ensure that all variables are in the scope of some commonizing quantifer. Consider the following example where we wish to introduce identity at some point in a derivation by means of the variable v, but there is no previous commonizing quantifier Л‖ on v.

```
  | ⋮
i | | Л‖v            C
  | |------------
  | | ⋮
j | | v = v          = I
  | | ⋮
  | ⋮
```

1.2.11 Identity Elimination (= E)

```
  | ⋮                                        | ⋮
  | ⋱                                        | ⋱
  | | ⋮                                      | | ⋮
i | | s = t                                i | | s = t
  | | ⋮                                      | | ⋮
j | | φ(..., s, ...)                       j | | φ(..., t, ...)
  | | ⋮                                      | | ⋮
k | | φ(..., t, ...)    = E i, j           k | | φ(..., s, ...)    = E i, j
  | | ⋮                                      | | ⋮
  | ⋮                                        | ⋮
```

Identity Elimination Condition: t is substitutable for s in φ (respectively, s is substitutable for t in φ).

2 Quantification Derivation Rules of QD^+

2.1 Derivation Rules QD

2.2 Inference Rules

2.2.1 Relabelling of Dummy Variables (RDV)

$$
\begin{array}{l|ll}
 & \vdots & \\
i & \forall v \varphi(v) & \\
 & \vdots & \\
j & \forall u \varphi(u) & \text{RDV i}
\end{array}
\qquad\qquad
\begin{array}{l|ll}
 & \vdots & \\
i & \exists v \varphi(v) & \\
 & \vdots & \\
j & \exists u \varphi(u) & \text{RDV i}
\end{array}
$$

Relabelling Dummy Variables Conditions:

1. u does not occur locally freely in $\varphi(v)$
2. u is substitutable for v in $\varphi(v)$.

2.2.2 Falsum Rules

$$
\begin{array}{l|l|ll}
 & \vdots & & \\
i & & \text{Л|}v & \\
\hline
 & & \vdots & \\
j & & \varphi & \\
 & & \vdots & \\
k & & \neg\varphi & \\
\ell & & \bot & \text{j, k} \\
m & \bot & & \\
 & \vdots & &
\end{array}
\qquad
\begin{array}{l|l|ll}
 & \vdots & & \\
 & & \vdots & \\
i & & \varphi & \mathcal{H} \\
\hline
 & & \vdots & \\
j & & \psi & \\
 & & \vdots & \\
k & & \neg\psi & \\
\ell & & \bot & \text{j, k} \\
m & \neg\varphi & & \neg\text{I} \\
 & \vdots & &
\end{array}
\qquad
\begin{array}{l|l|ll}
 & \vdots & & \\
 & & \vdots & \\
i & & \neg\varphi & \mathcal{H} \\
\hline
 & & \vdots & \\
j & & \psi & \\
 & & \vdots & \\
k & & \neg\psi & \\
\ell & & \bot & \text{j, k} \\
m & \varphi & & \neg\text{E} \\
 & \vdots & &
\end{array}
$$

2.3 Replacement Rules

2.3.1 Quantifier Negation (QN)

$$
\begin{aligned}
\forall v \varphi(v) &\vdash\dashv \neg\exists v \neg\varphi(v) \\
\exists v \varphi(v) &\vdash\dashv \neg\forall v \neg\varphi(v) \\
\neg\forall v \varphi(v) &\vdash\dashv \exists v \neg\varphi(v) \\
\neg\exists v \varphi(v) &\vdash\dashv \forall v \neg\varphi(v)
\end{aligned}
$$

2.3.2 Universal Quantifier Commutation (UQC)

$$
\begin{aligned}
\forall v \forall u \varphi(v, u) &\vdash\dashv \forall u \forall v \varphi(v, u) \\
\text{Л|}w_1 \ldots \text{Л|}w_k \forall v \forall u \varphi(v, u) &\vdash\dashv \text{Л|}w_1 \ldots \text{Л|}w_k \forall u \forall v \varphi(v, u) \\
\forall v \forall u \text{Л|}w_1 \ldots \text{Л|}w_k \varphi(v, u) &\vdash\dashv \forall u \forall v \text{Л|}w_1 \ldots \text{Л|}w_k \varphi(v, u)
\end{aligned}
$$

2.3.3 Existential Quantifier Commutation (EQC)

$$\exists v \exists u \varphi(v, u) \vdash\dashv \exists u \exists v \varphi(v, u)$$
$$\text{Л}\!| w_1 \ldots \text{Л}\!| w_k \exists v \exists u \varphi(v, u) \vdash\dashv \text{Л}\!| w_1 \ldots \text{Л}\!| w_k \exists u \exists v \varphi(v, u)$$
$$\exists v \exists u \text{Л}\!| w_1 \ldots \text{Л}\!| w_k \varphi(v, u) \vdash\dashv \exists u \exists v \text{Л}\!| w_1 \ldots \text{Л}\!| w_k \varphi(v, u)$$

2.3.4 Quantifier Distribution (QD)

$$\forall v(\varphi(v) \wedge \psi(v)) \vdash\dashv \forall v \varphi(v) \wedge \forall v \varphi(v)$$
$$\exists v(\varphi(v) \vee \psi(v)) \vdash\dashv \exists v \varphi(v) \vee \exists v \psi(v)$$

Suppose that $v \notin FV[\alpha]$.

$$\forall v(\alpha \wedge \varphi(v)) \vdash\dashv (\alpha \wedge \forall v \varphi(v))$$
$$\exists v(\alpha \wedge \varphi(v)) \vdash\dashv (\alpha \wedge \exists v \varphi(v))$$
$$\forall v(\alpha \vee \varphi(v)) \vdash\dashv (\alpha \vee \forall v \varphi(v))$$
$$\exists v(\alpha \vee \varphi(v)) \vdash\dashv (\alpha \vee \exists v \varphi(v))$$
$$\forall v(\alpha \implies \varphi(v)) \vdash\dashv (\alpha \implies \forall v \varphi(v))$$
$$\exists v(\alpha \implies \varphi(v)) \vdash\dashv (\alpha \implies \exists v \varphi(v))$$
$$\forall v(\varphi(v) \implies \alpha) \vdash\dashv (\exists v \varphi(v) \implies \alpha)$$
$$\exists v(\varphi(v) \implies \alpha) \vdash\dashv (\forall v \varphi(v) \implies \alpha)$$

2.3.5 Other Useful Quantifier Replacement Laws (QR)

$$\neg \forall v(\varphi(v) \implies \psi(v)) \vdash\dashv \exists v(\varphi(v) \wedge \neg \psi(v))$$
$$\neg \exists v(\varphi(v) \wedge \psi(v)) \vdash\dashv \forall v(\varphi(v) \implies \neg \psi(v))$$