

Philosophy 352: Derivation Rules for Sentential Logic

Dr. H. Korté
Department of Philosophy
University of Regina

January 16, 2012

Contents

1	Derivation Rules for Propositional Logic	2
1.1	The \wedge -Introduction and \wedge -Elimination Rules	2
1.1.1	\wedge -Introduction (\wedge I)	2
1.1.2	\wedge -Elimination (\wedge E)	2
1.2	The \implies -Introduction and \implies -Elimination Rules	2
1.2.1	\implies -Introduction (\implies I)	2
1.2.2	\implies -Elimination (\implies E)	3
1.3	The \neg -Introduction and \neg -Elimination Rules	3
1.3.1	\neg -Introduction (\neg I)	3
1.3.2	\neg -Elimination (\neg E)	3
1.4	The \vee -Introduction and \vee -Elimination Rules	4
1.4.1	\vee -Introduction (\vee I)	4
1.4.2	\vee -Elimination (\vee E)	4
1.5	The \iff -Introduction and \iff -Elimination Rules	5
1.5.1	\iff -Introduction (\iff I)	5
1.5.2	\iff -Elimination (\iff E)	5
1.6	The Reiteration Rule	6
2	Derivation Rules of SD^+	6
2.1	All the Derivation Rules of SD	6
2.2	Rules of Inference	6
2.2.1	Modus Tollens (MT)	6
2.2.2	Hypothetical Syllogism (HS)	6
2.2.3	Disjunctive Syllogism (DS)	7
2.3	Plus the Rules of Replacement	7
2.3.1	Commutation (Com)	7
2.3.2	Association (Assoc)	7
2.3.3	Implication (Impl)	7
2.3.4	Double Negation (DN)	7
2.3.5	De Morgan (DeM)	7
2.3.6	Idempotence (Idem)	7
2.3.7	Transposition (Trans)	7
2.3.8	Exportation (Exp)	7
2.3.9	Distribution (Dist)	8
2.3.10	Equivalence (Equiv)	8

1 Derivation Rules for Propositional Logic

In the following, the metavariables φ , ψ and χ are placeholders for any well-formed formula (wff) of the sentential language \mathcal{SL} .

1.1 The \wedge -Introduction and \wedge -Elimination Rules

1.1.1 \wedge -Introduction (\wedge I)

$$\begin{array}{c}
 \vdots \\
 i \mid \varphi \\
 \vdots \\
 j \mid \psi \\
 \vdots \\
 k \mid (\varphi \wedge \psi)
 \end{array}
 \quad \wedge \text{I } i, j
 \qquad
 \begin{array}{c}
 \vdots \\
 i \mid \varphi \\
 \vdots \\
 j \mid \psi \\
 \vdots \\
 k \mid (\psi \wedge \varphi)
 \end{array}
 \quad \wedge \text{I } i, j$$

Remark 1.1 The wffs of lines (i) and (j) may be interchanged, that is, φ and ψ may appear in a different order.

1.1.2 \wedge -Elimination (\wedge E)

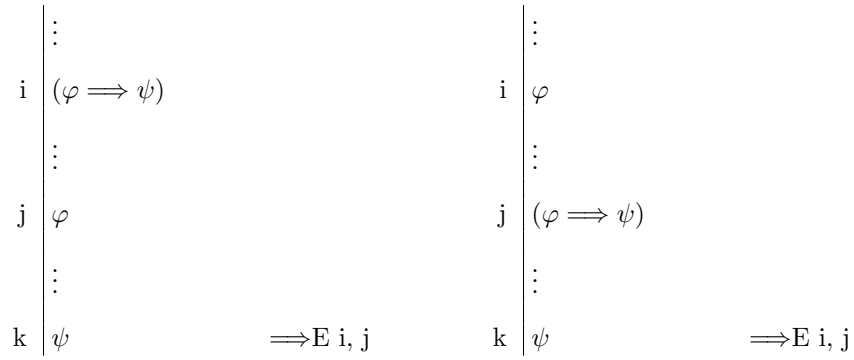
$$\begin{array}{c}
 \vdots \\
 i \mid (\varphi \wedge \psi) \\
 \vdots \\
 j \mid \varphi
 \end{array}
 \quad \wedge \text{E } i
 \qquad
 \begin{array}{c}
 \vdots \\
 i \mid (\varphi \wedge \psi) \\
 \vdots \\
 j \mid \psi
 \end{array}
 \quad \wedge \text{E } i$$

1.2 The \implies -Introduction and \implies -Elimination Rules

1.2.1 \implies -Introduction (\implies I)

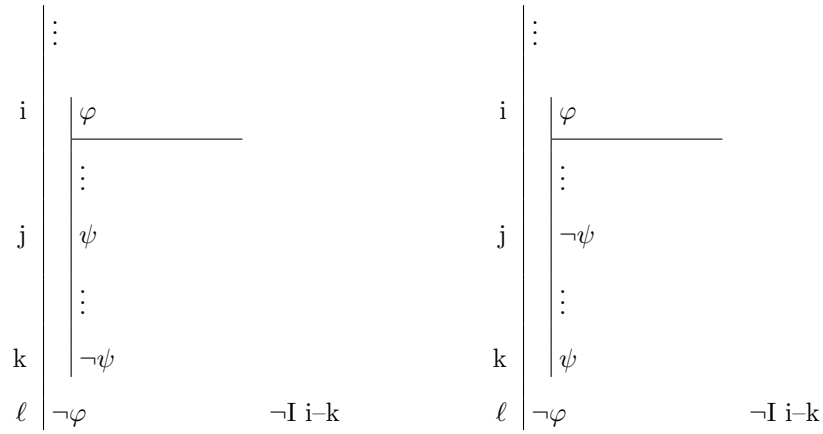
$$\begin{array}{c}
 i \mid \begin{array}{l} \varphi \\ \hline \vdots \end{array} \\
 j \mid \psi \\
 k \mid (\varphi \implies \psi)
 \end{array}
 \quad \begin{array}{l} \text{hyp} \\ \\ \implies \text{I } i-j \end{array}$$

1.2.2 \implies -Elimination (\implies E)

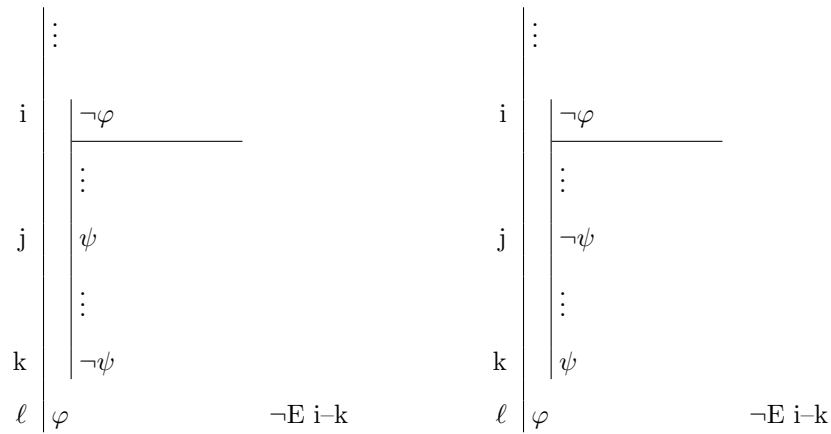


1.3 The \neg -Introduction and \neg -Elimination Rules

1.3.1 \neg -Introduction (\neg I)



1.3.2 \neg -Elimination (\neg E)



Remark 1.2 Since φ and ψ range over all wffs of SL the above rules of \neg I and \neg E hold for the special case when $\varphi = \psi$.

1.4 The \forall -Introduction and \forall -Elimination Rules

1.4.1 \forall -Introduction (\forall I)

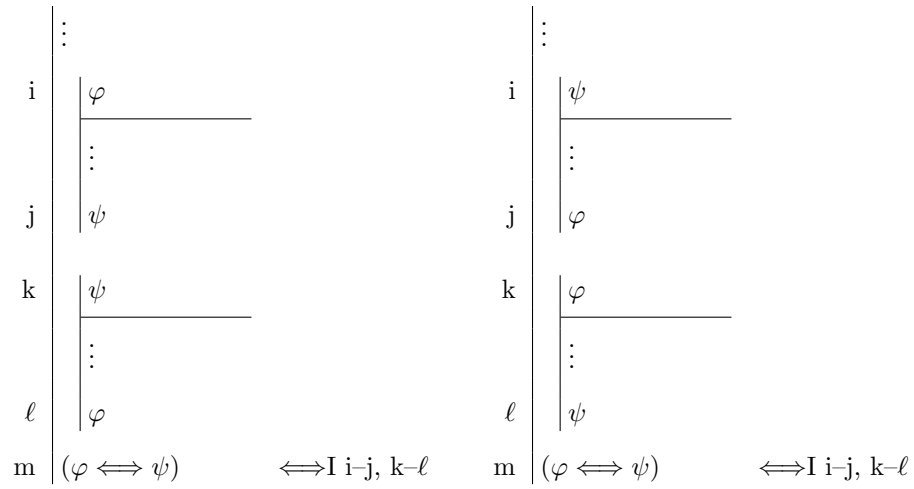
$$\begin{array}{c}
 \vdots \\
 i \mid \varphi \\
 \vdots \\
 j \mid (\varphi \vee \psi) \qquad \forall I \ i
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \\
 i \mid \varphi \\
 \vdots \\
 j \mid (\psi \vee \varphi) \qquad \forall I \ i
 \end{array}$$

1.4.2 \forall -Elimination (\forall E)

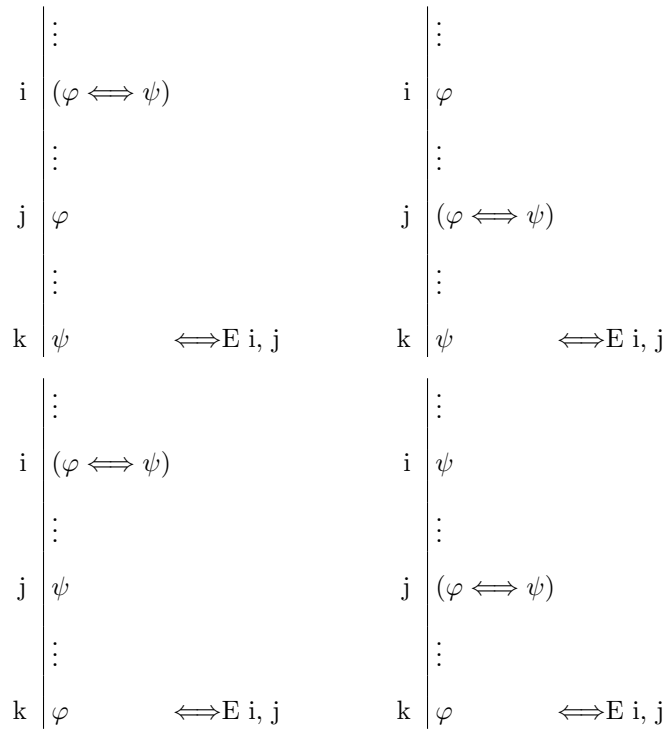
$$\begin{array}{c}
 \vdots \\
 i \mid (\varphi \vee \psi) \\
 \vdots \\
 j \mid \begin{array}{l} \varphi \\ \hline \vdots \\ \chi \end{array} \\
 \vdots \\
 \ell \mid \begin{array}{l} \psi \\ \hline \vdots \\ \chi \end{array} \\
 m \mid \chi \\
 n \mid \chi \qquad \forall E \ i, j-k, \ell-m
 \end{array}
 \qquad
 \begin{array}{c}
 \vdots \\
 i \mid (\varphi \vee \psi) \\
 \vdots \\
 j \mid \begin{array}{l} \psi \\ \hline \vdots \\ \chi \end{array} \\
 \vdots \\
 \ell \mid \begin{array}{l} \varphi \\ \hline \vdots \\ \chi \end{array} \\
 m \mid \chi \\
 n \mid \chi \qquad \forall E \ i, j-k, \ell-m
 \end{array}$$

1.5 The \iff -Introduction and \iff -Elimination Rules

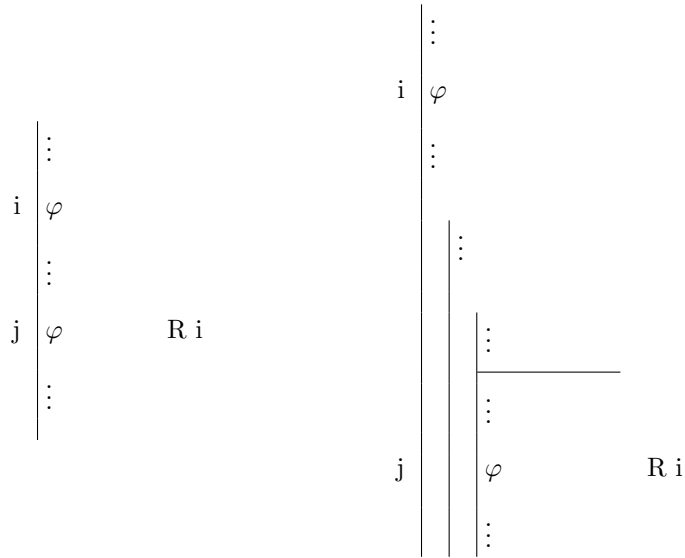
1.5.1 \iff -Introduction (\iff I)



1.5.2 \iff -Elimination (\iff E)



1.6 The Reiteration Rule



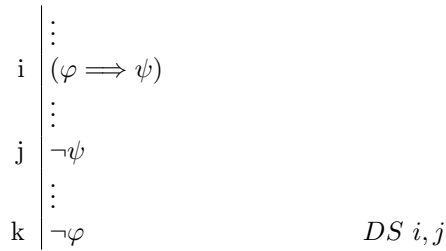
Note: One may **import**, that is, reiterate a wff (from outside to inside, that is, from left to right) across any number of scope lines, but one may **not export**, that is, reiterate a wff (from inside to outside, that is, from right to left).

2 Derivation Rules of SD^+

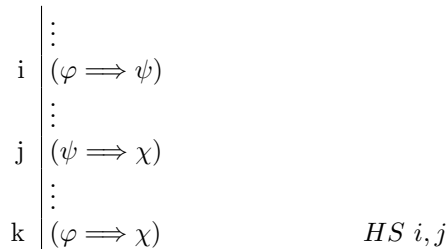
2.1 All the Derivation Rules of SD

2.2 Rules of Inference

2.2.1 Modus Tollens (MT)



2.2.2 Hypothetical Syllogism (HS)



2.2.3 Disjunctive Syllogism (DS)

$$\begin{array}{c}
 \vdots \\
 i \mid (\varphi \vee \psi) \\
 \vdots \\
 j \mid \neg\varphi \\
 \vdots \\
 k \mid \psi
 \end{array}
 \quad DS\ i,j
 \qquad
 \begin{array}{c}
 \vdots \\
 i \mid (\varphi \vee \psi) \\
 \vdots \\
 j \mid \neg\psi \\
 \vdots \\
 k \mid \varphi
 \end{array}
 \quad DS\ i,j$$

Remark 2.1 *It should be clear that the rules (MT), (HS) and (DS) hold when the wffs of lines (i) and (j) are interchanged.*

2.3 Plus the Rules of Replacement

2.3.1 Commutation (Com)

$$\begin{aligned}
 (\varphi \wedge \psi) &\vdash\vdash (\psi \wedge \varphi) \\
 (\varphi \vee \psi) &\vdash\vdash (\psi \vee \varphi)
 \end{aligned}$$

2.3.2 Association (Assoc)

$$\begin{aligned}
 (\varphi \wedge (\psi \wedge \chi)) &\vdash\vdash ((\varphi \wedge \psi) \wedge \chi) \\
 (\varphi \vee (\psi \vee \chi)) &\vdash\vdash ((\varphi \vee \psi) \vee \chi)
 \end{aligned}$$

2.3.3 Implication (Impl)

$$(\varphi \implies \psi) \vdash\vdash (\neg\varphi \vee \psi)$$

2.3.4 Double Negation (DN)

$$\psi \vdash\vdash \neg\neg\psi$$

2.3.5 De Morgan (DeM)

$$\begin{aligned}
 \neg(\varphi \wedge \psi) &\vdash\vdash (\neg\varphi \vee \neg\psi) \\
 \neg(\varphi \vee \psi) &\vdash\vdash (\neg\varphi \wedge \neg\psi)
 \end{aligned}$$

2.3.6 Idempotence (Idem)

$$\begin{aligned}
 \varphi &\vdash\vdash (\varphi \wedge \varphi) \\
 \varphi &\vdash\vdash (\varphi \vee \varphi)
 \end{aligned}$$

2.3.7 Transposition (Trans)

$$(\varphi \implies \psi) \vdash\vdash (\neg\psi \implies \neg\varphi)$$

2.3.8 Exportation (Exp)

$$(\varphi \implies (\psi \implies \chi)) \vdash\vdash ((\varphi \wedge \psi) \implies \chi)$$

2.3.9 Distribution (Dist)

$$\begin{aligned}(\varphi \wedge (\psi \vee \chi)) &\vdash\vdash ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \\(\varphi \vee (\psi \wedge \chi)) &\vdash\vdash ((\varphi \vee \psi) \wedge (\varphi \vee \chi))\end{aligned}$$

2.3.10 Equivalence (Equiv)

$$\begin{aligned}(\varphi \iff \psi) &\vdash\vdash ((\varphi \implies \psi) \wedge (\psi \implies \varphi)) \\(\varphi \iff \psi) &\vdash\vdash ((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))\end{aligned}$$

Remark 2.2 *Since the metavariables φ, ψ, χ range over all well-formed formulas of Sentential Logic \mathcal{SL} , the above rules holds for the special case when $\varphi = \psi = \chi$.*