Philosophy 352: Derivation Rules for Sentential Logic

Dr. H. Korté
Department of Philosophy
University of Regina

January 16, 2012

Contents

1 Derivation Rules for Propositional Logic

In the following, the metavariables φ, ψ and χ are placeholders for any well-formed formula (wff) of the sentential language $\mathcal{SL}$.

1.1 The $\wedge$-Introduction and $\wedge$-Elimination Rules

1.1.1 $\wedge$-Introduction ($\wedge$ I)

$$
\begin{array}{l|ll}
 & \vdots & \\
\text{i} & \varphi & \\
 & \vdots & \\
\text{j} & \psi & \\
 & \vdots & \\
\text{k} & (\varphi \wedge \psi) & \wedge\text{I i, j}
\end{array}
\qquad\qquad
\begin{array}{l|ll}
 & \vdots & \\
\text{i} & \varphi & \\
 & \vdots & \\
\text{j} & \psi & \\
 & \vdots & \\
\text{k} & (\psi \wedge \varphi) & \wedge\text{I i, j}
\end{array}
$$

Remark 1.1 *The wffs of lines (i) and (j) may be interchanged, that is, φ and ψ may appear in a different order.*

1.1.2 $\wedge$-Elimination ($\wedge$ E)

$$
\begin{array}{l|ll}
 & \vdots & \\
\text{i} & (\varphi \wedge \psi) & \\
 & \vdots & \\
\text{j} & \varphi & \wedge\text{E i}
\end{array}
\qquad\qquad
\begin{array}{l|ll}
 & \vdots & \\
\text{i} & (\varphi \wedge \psi) & \\
 & \vdots & \\
\text{j} & \psi & \wedge\text{E i}
\end{array}
$$

1.2 The $\Longrightarrow$-Introduction and $\Longrightarrow$-Elimination Rules

1.2.1 $\Longrightarrow$-Introduction ($\Longrightarrow$ I)

$$
\begin{array}{l|ll}
\text{i} & \quad \begin{array}{|l}\varphi \\ \hline \vdots \\ \psi\end{array} & \text{hyp} \\
\text{j} & & \\
\text{k} & (\varphi \Longrightarrow \psi) & \Longrightarrow\text{I i–j}
\end{array}
$$

1.2.2 $\Longrightarrow$-Elimination ($\Longrightarrow$ E)

	$\vdots$	
i	$(\varphi \Longrightarrow \psi)$	
	$\vdots$	
j	φ	
	$\vdots$	
k	ψ	$\Longrightarrow$E i, j

	$\vdots$	
i	φ	
	$\vdots$	
j	$(\varphi \Longrightarrow \psi)$	
	$\vdots$	
k	ψ	$\Longrightarrow$E i, j

1.3 The $\neg$-Introduction and $\neg$-Elimination Rules

1.3.1 $\neg$-Introduction ($\neg$ I)

	$\vdots$	
i	φ	
	$\vdots$	
j	ψ	
	$\vdots$	
k	$\neg\psi$	
ℓ	$\neg\varphi$	$\neg$I i–k

	$\vdots$	
i	φ	
	$\vdots$	
j	$\neg\psi$	
	$\vdots$	
k	ψ	
ℓ	$\neg\varphi$	$\neg$I i–k

1.3.2 $\neg$-Elimination ($\neg$ E)

	$\vdots$	
i	$\neg\varphi$	
	$\vdots$	
j	ψ	
	$\vdots$	
k	$\neg\psi$	
ℓ	φ	$\neg$E i–k

	$\vdots$	
i	$\neg\varphi$	
	$\vdots$	
j	$\neg\psi$	
	$\vdots$	
k	ψ	
ℓ	φ	$\neg$E i–k

Remark 1.2 *Since φ and ψ range over* **all** *wffs of $\mathcal{SL}$ the above rules of $\neg$ I and $\neg$ E hold for the special case when $\varphi = \psi$.*

1.4 The ∨-Introduction and ∨-Elimination Rules

1.4.1 ∨-Introduction (∨ I)

	$\vdots$	
i	φ	
	$\vdots$	
j	$(\varphi \vee \psi)$	∨I i

	$\vdots$	
i	φ	
	$\vdots$	
j	$(\psi \vee \varphi)$	∨I i

1.4.2 ∨-Elimination (∨ E)

	$\vdots$	
i	$(\varphi \vee \psi)$	
	$\vdots$	
j	| φ	
	| $\vdots$	
k	| χ	
	$\vdots$	
ℓ	| ψ	
	| $\vdots$	
m	| χ	
n	χ	∨E i, j–k, ℓ–m

	$\vdots$	
i	$(\varphi \vee \psi)$	
	$\vdots$	
j	| ψ	
	| $\vdots$	
k	| χ	
	$\vdots$	
ℓ	| φ	
	| $\vdots$	
m	| χ	
n	χ	∨E i, j–k, ℓ–m

1.5 The $\Longleftrightarrow$-Introduction and $\Longleftrightarrow$-Elimination Rules

1.5.1 $\Longleftrightarrow$-Introduction ($\Longleftrightarrow$ I)

	$\vdots$	
i	$\quad\vert\ \varphi$ (assumption)	
	$\quad\vert\ \vdots$	
j	$\quad\vert\ \psi$	
k	$\quad\vert\ \psi$ (assumption)	
	$\quad\vert\ \vdots$	
ℓ	$\quad\vert\ \varphi$	
m	$(\varphi \Longleftrightarrow \psi)$	$\Longleftrightarrow$I i–j, k–ℓ

	$\vdots$	
i	$\quad\vert\ \psi$ (assumption)	
	$\quad\vert\ \vdots$	
j	$\quad\vert\ \varphi$	
k	$\quad\vert\ \varphi$ (assumption)	
	$\quad\vert\ \vdots$	
ℓ	$\quad\vert\ \psi$	
m	$(\varphi \Longleftrightarrow \psi)$	$\Longleftrightarrow$I i–j, k–ℓ

1.5.2 $\Longleftrightarrow$-Elimination ($\Longleftrightarrow$ E)

	$\vdots$	
i	$(\varphi \Longleftrightarrow \psi)$	
	$\vdots$	
j	φ	
	$\vdots$	
k	ψ	$\Longleftrightarrow$E i, j

	$\vdots$	
i	φ	
	$\vdots$	
j	$(\varphi \Longleftrightarrow \psi)$	
	$\vdots$	
k	ψ	$\Longleftrightarrow$E i, j

	$\vdots$	
i	$(\varphi \Longleftrightarrow \psi)$	
	$\vdots$	
j	ψ	
	$\vdots$	
k	φ	$\Longleftrightarrow$E i, j

	$\vdots$	
i	ψ	
	$\vdots$	
j	$(\varphi \Longleftrightarrow \psi)$	
	$\vdots$	
k	φ	$\Longleftrightarrow$E i, j

1.6 The Reiteration Rule

	⋮	
i	φ	
	⋮	
j	φ	R i
	⋮	

	⋮	
i	φ	
	⋮	
	| ⋮	
	| | ⋮	
	| | ———	
	| | ⋮	
j	| | φ	R i
	| | ⋮	

Note: One may **import**, that is, reiterate a wff (from outside to inside, that is, from left to right) across any number of scope lines, but one may **not export**, that is, reiterate a wff (from inside to outside, that is, from right to left).

2 Derivation Rules of SD^+

2.1 All the Derivation Rules of SD

2.2 Rules of Inference

2.2.1 Modus Tollens (MT)

	⋮	
i	$(\varphi \Longrightarrow \psi)$	
	⋮	
j	$\neg\psi$	
	⋮	
k	$\neg\varphi$	$DS\ i, j$

2.2.2 Hypothetical Syllogism (HS)

	⋮	
i	$(\varphi \Longrightarrow \psi)$	
	⋮	
j	$(\psi \Longrightarrow \chi)$	
	⋮	
k	$(\varphi \Longrightarrow \chi)$	$HS\ i, j$

2.2.3 Disjunctive Syllogism (DS)

$$\begin{array}{r|ll} & \vdots & \\ \text{i} & (\varphi \vee \psi) & \\ & \vdots & \\ \text{j} & \neg\varphi & \\ & \vdots & \\ \text{k} & \psi & DS\ i,j \end{array} \qquad \begin{array}{r|ll} & \vdots & \\ \text{i} & (\varphi \vee \psi) & \\ & \vdots & \\ \text{j} & \neg\psi & \\ & \vdots & \\ \text{k} & \varphi & DS\ i,j \end{array}$$

Remark 2.1 *It should be clear that the rules (MT), (HS) and (DS) hold when the wffs of lines (i) and (j) are interchanged.*

2.3 Plus the Rules of Replacement

2.3.1 Commutation (Com)

$$(\varphi \wedge \psi) \vdash\dashv (\psi \wedge \varphi)$$
$$(\varphi \vee \psi) \vdash\dashv (\psi \vee \varphi)$$

2.3.2 Association (Assoc)

$$(\varphi \wedge (\psi \wedge \chi)) \vdash\dashv ((\varphi \wedge \psi) \wedge \chi)$$
$$(\varphi \vee (\psi \vee \chi)) \vdash\dashv ((\varphi \vee \psi) \vee \chi)$$

2.3.3 Implication (Impl)

$$(\varphi \Longrightarrow \psi) \vdash\dashv (\neg\varphi \vee \psi)$$

2.3.4 Double Negation (DN)

$$\psi \vdash\dashv \neg\neg\psi$$

2.3.5 De Morgan (DeM)

$$\neg(\varphi \wedge \psi) \vdash\dashv (\neg\varphi \vee \neg\psi)$$
$$\neg(\varphi \vee \psi) \vdash\dashv (\neg\varphi \wedge \neg\psi)$$

2.3.6 Idempotence (Idem)

$$\varphi \vdash\dashv (\varphi \wedge \varphi)$$
$$\varphi \vdash\dashv (\varphi \vee \varphi)$$

2.3.7 Transposition (Trans)

$$(\varphi \Longrightarrow \psi) \vdash\dashv (\neg\psi \Longrightarrow \neg\varphi)$$

2.3.8 Exportation (Exp)

$$(\varphi \Longrightarrow (\psi \Longrightarrow \chi)) \vdash\dashv ((\varphi \wedge \psi) \Longrightarrow \chi)$$

2.3.9 Distribution (Dist)

$$(\varphi \wedge (\psi \vee \chi)) \vdash\dashv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$$
$$(\varphi \vee (\psi \wedge \chi)) \vdash\dashv ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$$

2.3.10 Equivalence (Equiv)

$$(\varphi \iff \psi) \vdash\dashv ((\varphi \implies \psi) \wedge (\psi \implies \varphi))$$
$$(\varphi \iff \psi) \vdash\dashv ((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$$

Remark 2.2 *Since the metavariables φ, ψ, χ range over* **all** *well-formed formulas of Sentential Logic $\mathcal{SL}$, the above rules holds for the special case when $\varphi = \psi = \chi$.*